

Exotic symmetric space over a finite field, I

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ABSTRACT. Let V be a $2n$ -dimensional vector space over an algebraically closed field \mathbf{k} with $\text{ch } \mathbf{k} \neq 2$. Let $G = GL(V)$ and $H = Sp_{2n}$ be the symplectic group obtained as $H = G^\theta$ for an involution θ on G . We also denote by θ the induced involution on $\mathfrak{g} = \text{Lie } G$. Consider the variety $G/H \times V$ on which H acts naturally. Let $\mathfrak{g}_{\text{nil}}^{-\theta}$ be the set of nilpotent elements in the -1 eigenspace of θ in \mathfrak{g} . The role of the unipotent variety for G in our setup is played by $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$, which coincides with Kato's exotic nilpotent cone. Kato established, in the case where $\mathbf{k} = \mathbf{C}$, the Springer correspondence between the set of irreducible representations of the Weyl group of type C_n and the set of H -orbits in $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$ by applying Ginzburg theory for affine Hecke algebras. In this paper we develop a theory of character sheaves on $G/H \times V$, and give an alternate proof for Kato's result on the Springer correspondence based on the theory of character sheaves.

INTRODUCTION

Let $G' = GL_n$ acting on the n -dimensional vector space V' over an algebraically closed field \mathbf{k} , and $\mathfrak{g}' = \text{Lie } G'$. Let G'_{uni} (resp. $\mathfrak{g}'_{\text{nil}}$) be the unipotent variety of G' (resp. the nilpotent cone of \mathfrak{g}'). We consider the action of G' on the variety $\mathfrak{g}'_{\text{nil}} \times V'$, where G' acts on $\mathfrak{g}'_{\text{nil}}$ by the adjoint action, and on V' by the natural action. By Achar-Henderson [AH] and Travkin [T], $\mathfrak{g}'_{\text{nil}} \times V'$ has a finitely many G' -orbits parametrized by double partitions of n . Following [AH], we call $\mathfrak{g}'_{\text{nil}} \times V'$ the enhanced nilpotent cone. In [AH], they studied the intersection cohomology of the closure of such orbits, and showed that associated Poincaré polynomials give Kostka polynomials labelled by double partitions, introduced in [S2], which is an analogy of the classical result by Lusztig [L1] relating nilpotent orbits in $\mathfrak{g}_{\text{nil}}$ and Kostka polynomials. Passing to the group case, we consider the action of G' on the variety $G' \times V'$, where G' acts on G' by conjugation, and on V' by the natural action. Finkelberg-Ginzburg-Travkin [FGT] constructed a family of G' -equivariant simple perverse sheaves on $G' \times V'$, and developed an analogy of the theory of character sheaves on G' , where $G'_{\text{uni}} \times V' \simeq \mathfrak{g}'_{\text{nil}} \times V'$ plays a role of the unipotent variety of G' . They conjecture that the characteristic functions of such character sheaves on $G' \times V'$ provide a basis of the space of $G'(\mathbf{F}_q)$ -invariant functions on $(G' \times V')(\mathbf{F}_q)$.

Assume that \mathbf{k} is an algebraic closure of a finite field \mathbf{F}_q with $\text{ch } \mathbf{k} \neq 2$, and let V be a $2n$ -dimensional vector space over \mathbf{k} . Let $H = Sp_{2n}$ be the symplectic group obtained as the fixed point subgroup G^θ for an involutive automorphism

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θ on $G = GL(V)$, and consider the symmetric space G/H . In [BKS], Bannai-Kawanaka-Song studied the characters of the Hecke algebra $\mathcal{H} = \mathcal{H}(G(\mathbf{F}_q), H(\mathbf{F}_q))$ associated to the pair $H(\mathbf{F}_q) \subset G(\mathbf{F}_q)$, and showed that the character table of \mathcal{H} is basically obtained from the character table of $GL_n(\mathbf{F}_q)$ by replacing q by q^2 in an appropriate sense. On the other hand, in [H1] Henderson tried to reconstruct the result of [BKS] in terms of the geometry of the symmetric space G/H . Let $\mathfrak{g}_{\text{nil}}^{-\theta} = \{g \in \mathfrak{g}_{\text{nil}} \mid \theta(g) = -g\}$ for the involution θ induced on $\mathfrak{g} = \text{Lie } G$. Then H acts on $\mathfrak{g}_{\text{nil}}^{-\theta}$, and H -orbits are labelled by partitions of n . He showed, in particular, that Poincaré polynomials associated to the intersection cohomology of the closure of those orbits provide Kostka polynomials, replacing the variable q by q^2 , which is a geometric counter part of the result of [BKS].

In this paper, we consider the variety $G/H \times V$ as a generalization of above two cases. H acts on $G/H \times V$ as a left multiplication on G/H , and as the natural action on V . In this setup, the role of the unipotent variety for G is played by the variety $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$, which is nothing but the exotic nilpotent cone introduced by Kato [Ka1]. So we shall call $G/H \times V$ the exotic symmetric space. H acts on $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$. Kato showed that the number of H -orbits is finite and they are parametrized by double partitions of n (a reformulation by Achar-Henderson [AH]). It is expected an interesting relationship between the intersection cohomology of the closure of those orbits and Kostka polynomials labelled by double partitions. Our aim is to construct a theory of character sheaves on $G/H \times V$ as an analogy of the theory for G' and $G' \times V'$. In fact, in [HT] Henderson-Trapa propose a construction of character sheaves on $G/H \times V$, as a natural generalization of mirabolic character sheaves due to [FGT], “exotic character sheaves” in their terminology. In their framework, the character sheaves constructed in this paper just cover the principal series part. However, we expect that any exotic character sheaf can be obtained by our construction.

The main result in this paper is the Springer correspondence between the set of irreducible representations of the Weyl group of type C_n and the set of H -orbits of $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$ through the intersection cohomology of the closure of H -orbits. In fact, the Springer correspondence for exotic nilcone was first established by [Ka1], by using Ginzburg theory of affine Hecke algebras. In [Ka2], he determined the correspondence explicitly by computing Joseph polynomials associated to H -orbits. So our result gives an alternate approach to Kato’s result based on the theory of character sheaves, which is quite similar to the original proof of the Springer correspondence due to Borho-MacPherson [BM]. We prove the restriction theorem for Springer representations, which is an analogy of Lusztig’s restriction theorem [L2] with respect to the generalized Springer correspondence, and we determine the correspondence explicitly by using this restriction theorem.

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Contents

1. Symmetric space GL_{2n}/Sp_{2n}
2. H -orbits on $G_{\text{reg}}^{\theta} \times V$

3. Intersection cohomology on $G_{\text{reg}}^{\iota\theta} \times V$
4. Intersection cohomology on $G^{\iota\theta} \times V$
5. Springer correspondence
6. Restriction theorem
7. Determination of Springer correspondence

1. SYMMETRIC SPACE GL_{2n}/Sp_{2n}

1.1. Let \mathbf{k} be an algebraic closure of a finite field \mathbf{F}_q with $\text{char } \mathbf{k} \neq 2$. Let V be a $2n$ dimensional vector space over \mathbf{k} , with basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$. Let $G = GL_{2n}$ and $\mathfrak{g} = \text{Lie } G = \mathfrak{gl}_{2n}$. Consider an involutive automorphism $\theta : G \rightarrow G$ given by

$$\theta(g) = J^{-1}({}^t g^{-1})J \quad \text{with } J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

where 1_n is the identity matrix of degree n , and put $H = G^\theta$. Then H is the symplectic group Sp_{2n} with respect to the symplectic form $\langle u, v \rangle = {}^t u J v$ for $u, v \in V$ under the identification $V \simeq \mathbf{k}^{2n}$ via the basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$, which gives rise to a symplectic basis. We denote by the same symbol θ the involution induced on \mathfrak{g} . Then $\theta(x) = -J^{-1}({}^t x)J$ for $x \in \mathfrak{g}$. We have a decomposition $\mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{g}^{-\theta}$, where

$$\mathfrak{g}^{\pm\theta} = \{x \in \mathfrak{g} \mid \theta(x) = \pm x\}.$$

Let x^* be the adjoint of $x \in \mathfrak{g}$ with respect to the form $\langle \cdot, \cdot \rangle$. Then we have $x^* = J^{-1}({}^t x)J$, and so $\mathfrak{g}^{\pm\theta} = \{x \in \mathfrak{g} \mid x^* = \mp x\}$. In particular, $\mathfrak{g}^{-\theta}$ coincides with the set of self-adjoint matrices in \mathfrak{gl}_{2n} .

1.2. Let $\iota : G \rightarrow G$ be the anti-automorphism $g \mapsto g^{-1}$. We consider the set $G^{\iota\theta} = \{g \in G \mid \theta(g) = g^{-1}\}$. Then as in the Lie algebra case, $G^{\iota\theta}$ coincides with the set of non-degenerate self-adjoint matrices. In particular, it is connected. Let B be the subgroup of G consisting of the elements of the form

$$\begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix},$$

where b_1, b_2, c are square matrices of degree n , with b_1 upper triangular and b_2 lower triangular. Let T be the set of all diagonal matrices in G . Then B is a Borel subgroup of G containing T , and B and T are both θ -stable. We have

$$(1.2.1) \quad T^\theta = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in D_n \right\}, \quad T^{\iota\theta} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in D_n \right\},$$

where D_n is the set of diagonal matrices in GL_n . Moreover we have

$$(1.2.2) \quad \begin{aligned} B^\theta &= \{g = \begin{pmatrix} b & c \\ 0 & {}^t b^{-1} \end{pmatrix} \in B \mid {}^t c = b^{-1} c {}^t b\}, \\ B^{\iota\theta} &= \{g = \begin{pmatrix} b & c \\ 0 & {}^t b \end{pmatrix} \in B \mid {}^t c = -c\}. \end{aligned}$$

We note that

$$(1.2.3) \quad G^{\iota\theta} = \{g\theta(g)^{-1} \mid g \in G\}.$$

In fact, it is known by a general theory ([R, 2.2], see also [Gi, 3.3]) that the right hand side of (1.2.3) coincides with the connected component of $G^{\iota\theta}$. Since $G^{\iota\theta}$ is connected, (1.2.3) holds. It is also checked directly as follows; take $x \in G^{\iota\theta}$. Then x is self-adjoint, and so there exists an isotropic flag $(V_1 \subset V_2 \subset \cdots \subset V_n)$ in V stable by x . Since $G^{\iota\theta}$ is H -stable, by replacing x by its H -conjugate, we may assume that $x \in B^{\iota\theta}$. We write $x = \begin{pmatrix} b & c \\ 0 & {}^t b \end{pmatrix}$ as in (1.2.2). If we put $y = \begin{pmatrix} b & 0 \\ 0 & 1_n \end{pmatrix}$, then $\theta(y) = \begin{pmatrix} 1_n & 0 \\ 0 & {}^t b^{-1} \end{pmatrix}$, and so $y^{-1}x\theta(y) = \begin{pmatrix} 1_n & c' \\ 0 & 1_n \end{pmatrix} \in B^{\iota\theta}$. Since c' is a skew-symmetric matrix, one can write $c' = a - {}^t a$ for a square matrix a of degree n . Then we have

$$\begin{pmatrix} 1_n & c' \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & -{}^t a \\ 0 & 1_n \end{pmatrix} = z\theta(z)^{-1}$$

with $z = \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix}$. This implies that $x = g\theta(g)^{-1}$ for some $g \in G$. The opposite inclusion in (1.2.3) is clear.

Note that the above argument shows, in particular, that

$$(1.2.4) \quad B^{\iota\theta} = \{b\theta(b)^{-1} \mid b \in B\}.$$

Remark 1.3. The following properties hold.

(i) If B' is a θ -stable Borel subgroup of G , then B and B' are conjugate under H .

(ii) If B' is a θ -stable Borel subgroup containing a θ -stable maximal torus T' , then the pairs (B', T') and (B, T) are conjugate under H .

(iii) A θ -stable torus S of G is called θ -anisotropic if $\theta(s) = s^{-1}$ for any $s \in S$. Then $T^{\iota\theta}$ is a maximal θ -anisotropic torus, and any maximal torus of G containing a maximal θ -anisotropic torus is conjugate to T under H .

In fact, for (i), B' is written as $B' = xBx^{-1}$ for some $x \in G$. Then we have $x^{-1}\theta(x) \in B^{\iota\theta}$. By (1.2.4), there exists $b \in B$ such that $b\theta(b)^{-1} = x^{-1}\theta(x)$, and $g = xb \in H$. Then $gBg^{-1} = B'$. For (ii), assume that $xBx^{-1} = B'$, $xTx^{-1} = T'$. Then $x^{-1}\theta(x) \in B^{\iota\theta} \cap N_G(T) = T^{\iota\theta}$, and there exists $t \in T$ such that $t\theta(t)^{-1} = x^{-1}\theta(x)$. Then $g = xt \in H$, and we have $g(B, T)g^{-1} = (B', T')$. For (iii), it is clear that $T^{\iota\theta}$ is maximal θ -anisotropic. By a general theory ([V], see also [R, 2.7]) that

maximal θ -anisotropic tori are conjugate under H . Since $Z_G(T^{\iota\theta}) = Z_H(T^{\iota\theta}) \cdot T^{\iota\theta}$, (iii) follows. Note that (iii) implies that any maximal torus containing a maximal θ -anisotropic torus is θ -stable.

1.4. By (1.2.3), the symmetric space G/H can be identified with $G^{\iota\theta}$ so that the natural map $\pi : G \rightarrow G/H$ is given by $\pi : G \rightarrow G^{\iota\theta}, g \mapsto g\theta(g)^{-1}$ (cf. [R, Lemma 2.4]). Under this identification, the left multiplication of H on G/H turns out to be the conjugation action of H on $G^{\iota\theta}$. Let A be a closed subgroup of G isomorphic to GL_n defined by

$$A = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1_n \end{pmatrix} \in G \mid x \in GL_n \right\}.$$

Then for $a = \begin{pmatrix} x & 0 \\ 0 & 1_n \end{pmatrix} \in A$, we have $a\theta(a)^{-1} = \begin{pmatrix} x & 0 \\ 0 & {}^t x \end{pmatrix} \in G^{\iota\theta}$.

We define a subgroup L of G by

$$L = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in G \mid x, y \in GL_n \right\} \simeq GL_n \times GL_n.$$

Then L is θ -stable, and $L^\theta = \{a\theta(a) \mid a \in A\}$, $L^{\iota\theta} = \{a\theta(a)^{-1} \mid a \in A\}$. The following lemma was proved by Klyachko [Kl] (cf. [BKS, Lemma 2.3.4]).

Lemma 1.5. (i) *For $a, a' \in A$, $a\theta(a)^{-1}$ is H -conjugate to $a'\theta(a')^{-1}$ if and only if a and a' are conjugate in A .*
(ii) *The map $a \mapsto a\theta(a)^{-1}$ induces a bijection between the set of conjugacy classes in A and the set of H -conjugacy classes in $G^{\iota\theta}$.*

1.6. We note that for any $g \in G^{\iota\theta}$, $Z_H(g)$ is a connected subgroup of H . In fact, as in remarked in [BKS, Lemma 2.2.1], for $x \in G$, $Z_G(x\theta(x)^{-1})$ is θ -stable, and we have $Z_H(x\theta(x)^{-1}) = (Z_G(x\theta(x)^{-1}))^\theta$. Since $Z_G(x\theta(x)^{-1})$ is a product of general linear groups, on which θ acts as an involutive automorphism, $Z_H(x\theta(x)^{-1})$ is a product of general linear groups and symplectic groups. Thus $Z_H(x\theta(x)^{-1})$ is connected.

1.7. Let $g = su = us$ be the Jordan decomposition of $g \in G$, where u is unipotent and s is semisimple. If $g \in G^{\iota\theta}$, then $s, u \in G^{\iota\theta}$, and the Jordan decomposition makes sense in $G^{\iota\theta}$. We denote by $G_{\text{uni}}^{\iota\theta}$ the set of unipotent elements in $G^{\iota\theta}$. Similarly, we have the Jordan decomposition of $\mathfrak{g}^{-\theta}$, and denote by $\mathfrak{g}_{\text{nil}}^{-\theta}$ the set of nilpotent elements in $\mathfrak{g}^{-\theta}$.

We define a map $\log : G^{\iota\theta} \rightarrow \mathfrak{g}^{-\theta}$ by the composite of the maps

$$\log : G^{\iota\theta} \xrightarrow{i} G \xrightarrow{j} \mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{g}^{-\theta} \xrightarrow{p_2} \mathfrak{g}^{-\theta},$$

where i is the inclusion map, and j is the map defined by $j(g) = g - 1$, and p_2 is the projection onto the second factor. Then \log is an H -equivariant morphism from $G^{\iota\theta}$ to $\mathfrak{g}^{-\theta}$ and $\log(1) = 0$, and $d\log_e : \mathfrak{g}^{-\theta} \rightarrow \mathfrak{g}^{-\theta}$ induces the identity map on $\mathfrak{g}^{-\theta}$. By Bradsley-Richardson [BR, Proposition 10.1], we see that the restriction of \log on $G_{\text{uni}}^{\iota\theta}$ gives rise to an H -equivariant isomorphism $G_{\text{uni}}^{\iota\theta} \simeq \mathfrak{g}_{\text{nil}}^{-\theta}$.

Under the correspondence in Lemma 1.5, the set of unipotent classes in A is mapped to the set of unipotent classes in $G^{\iota\theta}$. Since the set of unipotent classes in A is parametrized by the set \mathcal{P}_n of partitions of n , we see that

$$(1.7.1) \quad G_{\text{uni}}^{\iota\theta} / \sim_H \simeq \mathfrak{g}_{\text{nil}}^{-\theta} / \sim_H \simeq \mathcal{P}_n,$$

where X / \sim_H denotes the set of H -orbits of the G -variety X .

We consider the varieties $G_{\text{uni}}^{\iota\theta} \times V$ and $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$, with diagonal actions of H . Note that $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$ is nothing but the exotic nilpotent cone introduced by Kato [Ka1]. Now $\log \times \text{id}$ induces an H -equivariant isomorphism between $G_{\text{uni}}^{\iota\theta} \times V$ and $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$. By [Ka1], we know that $(\mathfrak{g}_{\text{nil}}^{-\theta} \times V) / \sim_H$ is parametrized by the set $\mathcal{P}_{n,2}$ of double partitions of n . Hence we have

$$(1.7.2) \quad (G_{\text{uni}}^{\iota\theta} \times V) / \sim_H \simeq (\mathfrak{g}_{\text{nil}}^{-\theta} \times V) / \sim_H \simeq \mathcal{P}_{n,2}.$$

We denote by \mathcal{O}_{λ} the H -orbit of $G_{\text{uni}}^{\iota\theta} \times V$ corresponding to $\lambda \in \mathcal{P}_{n,2}$. (Here we follow the parametrization given in [AH, Theorem 6.1] in connection with the parametrization of the orbits in the enhanced nilpotent cone.)

The closure relations of orbits in $G_{\text{uni}}^{\iota\theta} \times V$ are described as follows. For $\lambda = (\mu, \nu) \in \mathcal{P}_{n,2}$ with $\mu = (\mu_i), \nu = (\nu_i)$, we define a composition $c(\lambda)$ of n by $c(\lambda) = (\mu_1, \nu_1, \mu_2, \nu_2, \dots)$. We define a partial order on $\mathcal{P}_{n,2}$ by $\mu \leq \lambda$ if and only if $c(\mu) \leq c(\lambda)$, where the latter is the dominance order of compositions of n . Then we have by [AH, Theorem 6.3],

$$(1.7.3) \quad \mathcal{O}_{\mu} \subset \overline{\mathcal{O}_{\lambda}} \text{ if and only if } \mu \leq \lambda.$$

1.8. Let $T_{\text{reg}}^{\iota\theta}$ be the set of regular semisimple elements in $T^{\iota\theta}$, i.e., the set of semisimple elements in $T^{\iota\theta}$ such that all the eigenspaces on V have dimension 2. We define $G_{\text{reg}}^{\iota\theta} = \bigcup_{g \in H} g T_{\text{reg}}^{\iota\theta} g^{-1}$, the set of regular semisimple elements in $G^{\iota\theta}$. Then $G_{\text{reg}}^{\iota\theta}$ is open dense in $G^{\iota\theta}$. Take $t \in T_{\text{reg}}^{\iota\theta}$. Then $Z_H(t) = Z_H(T^{\iota\theta}) \simeq SL_2 \times \cdots \times SL_2$ (n copies). For the Borel subgroup B^{θ} of H , put

$$\tilde{G}^{\iota\theta} = \{(x, gB^{\theta}) \in G^{\iota\theta} \times H/B^{\theta} \mid g^{-1}xg \in B^{\iota\theta}\}$$

and define a map $\pi : \tilde{G}^{\iota\theta} \rightarrow G^{\iota\theta}$ by $\pi(x, gB^{\theta}) = x$. Then $\tilde{G}^{\iota\theta} \simeq H \times^{B^{\theta}} B^{\iota\theta}$ is a locally trivial fibration over H/B^{θ} with fibre isomorphic to $B^{\iota\theta}$ (the associated bundle of the principal B^{θ} -bundle $H \rightarrow H/B^{\theta}$). Since $B^{\iota\theta}$ is smooth by (1.2.2), $\tilde{G}^{\iota\theta}$ is smooth and irreducible. Moreover, π is proper.

We consider the pull-back $\pi^{-1}(G_{\text{reg}}^{\iota\theta})$ of $G_{\text{reg}}^{\iota\theta}$ under the map π , and let $\psi : \pi^{-1}(G_{\text{reg}}^{\iota\theta}) \rightarrow G_{\text{reg}}^{\iota\theta}$ be the restriction of π on $\pi^{-1}(G_{\text{reg}}^{\iota\theta})$. Then we have

$$\pi^{-1}(G_{\text{reg}}^{\iota\theta}) \simeq H \times^{B^{\theta}} B_{\text{reg}}^{\iota\theta} \simeq H \times^{(B^{\theta} \cap Z_H(T^{\iota\theta}))} T_{\text{reg}}^{\iota\theta},$$

where $B_{\text{reg}}^{\iota\theta} = B^{\iota\theta} \cap G_{\text{reg}}^{\iota\theta}$. Note that $B^{\theta} \cap Z_H(T^{\iota\theta})$ is a Borel subgroup of $Z_H(T^{\iota\theta}) \simeq SL_2 \times \cdots \times SL_2$, hence is of the form $B_2 \times \cdots \times B_2$ with a Borel subgroup B_2 of

SL_2 . Now the map ψ factors through $H \times^{Z_H(T^{\iota\theta})} T^{\iota\theta}$ as follows;

$$(1.8.1) \quad \psi : \pi^{-1}(G_{\text{reg}}^{\iota\theta}) \xrightarrow{\xi} H \times^{Z_H(T^{\iota\theta})} T_{\text{reg}}^{\iota\theta} \xrightarrow{\eta} G_{\text{reg}}^{\iota\theta},$$

where ξ is proper, and is a locally trivial fibration with fibre isomorphic to

$$Z_H(T^{\iota\theta})/(Z_H(T^{\iota\theta}) \cap B^\theta) \simeq (SL_2/B_2)^n \simeq \mathbf{P}_1^n,$$

and η is a finite Galois covering with group $\mathcal{W} = N_H(T^{\iota\theta})/Z_H(T^{\iota\theta}) \simeq S_n$ through the identifications

$$H \times^{Z_H(T^{\iota\theta})} T_{\text{reg}}^{\iota\theta} \simeq H/Z_H(T^{\iota\theta}) \times T_{\text{reg}}^{\iota\theta} \rightarrow (H/Z_H(T^{\iota\theta}) \times T_{\text{reg}}^{\iota\theta})/S_n \simeq G_{\text{reg}}^{\iota\theta},$$

where S_n acts on $H/Z_H(T^{\iota\theta}) \times T_{\text{reg}}^{\iota\theta}$ by $(w, (gZ_H(T^{\iota\theta}), t)) \mapsto (gw^{-1}Z_H(T^{\iota\theta}), wtw^{-1})$.

Summing up the above computation, we obtain the following lemma.

Lemma 1.9. *Let U be the unipotent radical of B . Then*

- (i) $\dim U^\theta = n^2$, $\dim B^\theta = n^2 + n$, $\dim U^{\iota\theta} = n^2 - n$, $\dim B^{\iota\theta} = n^2$.
- (ii) $\tilde{G}^{\iota\theta}$ is a smooth irreducible variety with $\dim \tilde{G}^{\iota\theta} = 2n^2$. π is a proper surjective map from $\tilde{G}^{\iota\theta}$ onto $G^{\iota\theta}$, and $\dim G^{\iota\theta} = 2n^2 - n$.

Proof. For (i), the statement for B^θ, U^θ is well-known. The statement for $B^{\iota\theta}, U^{\iota\theta}$ comes from (1.2.2). For (ii), $\dim \tilde{G}^{\iota\theta} = \dim H/B^\theta + \dim B^{\iota\theta} = \dim U^\theta + \dim B^{\iota\theta} = 2n^2$ by (i). By 1.4, $\dim G^{\iota\theta} = \dim G - \dim H = 2n^2 - n$. \square

1.10. We define a variety

$$\tilde{G}_{\text{uni}}^{\iota\theta} = \{(x, gB^\theta) \in G_{\text{uni}}^{\iota\theta} \times H/B^\theta \mid g^{-1}xg \in U^{\iota\theta}\}$$

and the map $\pi_1 : \tilde{G}_{\text{uni}}^{\iota\theta} \rightarrow G_{\text{uni}}^{\iota\theta}$ by $\pi_1(x, gB^\theta) \mapsto x$. Then $\tilde{G}_{\text{uni}}^{\iota\theta} \simeq H \times^{B^\theta} U^{\iota\theta}$ is a vector bundle over H/B^θ with fibre isomorphic to $U^{\iota\theta}$. Hence $\tilde{G}_{\text{uni}}^{\iota\theta}$ is smooth and irreducible, and π_1 is a surjective map onto $G_{\text{uni}}^{\iota\theta}$. For $x \in G_{\text{uni}}^{\iota\theta}$, we consider $\pi_1^{-1}(x) \simeq \{gB^\theta \in H/B^\theta \mid g^{-1}xg \in U^{\iota\theta}\}$. We have the following lemma. (i) and (ii) are proved in Lemma 5.11 and (2.3.11) in [BKS]. (iii) is immediate from (ii).

Lemma 1.11. *Assume that $x = y\theta(y)^{-1} \in G_{\text{uni}}^{\iota\theta}$ with $y \in A_{\text{uni}}$. Then*

- (i) $\dim \pi_1^{-1}(x) = (\dim Z_H(x) - \text{rank } H)/2$.
- (ii) $\dim H - \dim Z_H(x) = 2(\dim A - \dim Z_A(y))$.
- (iii) *Let $y = y_\lambda \in A$ be an element corresponding to $\lambda \in \mathcal{P}_n$. Put $x_\lambda = x$ and let \mathcal{O}_λ be the H -orbit of x_λ in $G_{\text{uni}}^{\iota\theta}$. Then we have $\dim \mathcal{O}_\lambda = 2n^2 - 2n - 4n(\lambda)$. In particular, $\dim \pi_1^{-1}(x) = 2n(\lambda) + n$.*

By using this, one can show that

Lemma 1.12. (i) $\tilde{G}_{\text{uni}}^{\iota\theta}$ is smooth and irreducible with $\dim \tilde{G}_{\text{uni}}^{\iota\theta} = 2n^2 - n$.
(ii) $\dim G_{\text{uni}}^{\iota\theta} = 2 \dim U^{\iota\theta} = 2n^2 - 2n$.

Proof. By 1.10, we have $\dim \tilde{G}_{\text{uni}}^{\iota\theta} = \dim H/B^\theta + \dim U^{\iota\theta} = \dim U^\theta + \dim U^{\iota\theta} = 2n^2 - n$. Hence (i) holds. We show (ii). Let y be a regular unipotent element in A , and put $x = y\theta(y)^{-1}$. Since $\dim Z_A(y) = n$, we have $\dim Z_H(x) = 3n$ by Lemma 1.11 (ii). Hence $\dim \psi_1^{-1}(x) = n$ by Lemma 1.11 (i). Since the H -orbit of x is open dense in $G_{\text{uni}}^{\iota\theta}$, we see that $\dim G_{\text{uni}}^{\iota\theta} = \dim \tilde{G}_{\text{uni}}^{\iota\theta} - n$. Hence (ii) holds. \square

1.13. We consider the direct image complex $\psi_* \bar{\mathbf{Q}}_l$ of the constant sheaf $\bar{\mathbf{Q}}_l$ on $\pi^{-1}(\tilde{G}_{\text{reg}}^{\iota\theta})$. In the notation in 1.8, since ξ is a locally trivial fibration with fibre isomorphic to \mathbf{P}_1^n , we see that

$$\xi_* \bar{\mathbf{Q}}_l \simeq H^\bullet(\mathbf{P}_1^n) \otimes \bar{\mathbf{Q}}_l \simeq \bigoplus_{J \subseteq [1, n]} \bar{\mathbf{Q}}_l[-2|J|],$$

where $J \subseteq [1, n]$ means that J runs over all the subsets of the set $[1, n] = \{1, 2, \dots, n\}$, and $H^\bullet(\mathbf{P}_1^n)$ denotes the graded space $\bigoplus_i H^{2i}(\mathbf{P}_1^n, \bar{\mathbf{Q}}_l)$ which we regard as a complex $\bigoplus_{J \subseteq [1, n]} \bar{\mathbf{Q}}_l[-2|J|]$. On the other hand, since η is a finite Galois covering with group S_n , we have $\text{End}(\eta_* \bar{\mathbf{Q}}_l) \simeq \bar{\mathbf{Q}}_l[S_n]$, the group algebra of S_n over $\bar{\mathbf{Q}}_l$. Thus $\eta_* \bar{\mathbf{Q}}_l$ is decomposed as

$$\eta_* \bar{\mathbf{Q}}_l \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V_\lambda \otimes \mathcal{L}_\lambda,$$

where V_λ is the irreducible S_n -module corresponding to the partition $\lambda \in \mathcal{P}_n$, and $\mathcal{L}_\lambda = \text{Hom}_{S_n}(V_\lambda, \eta_* \bar{\mathbf{Q}}_l)$ is an irreducible local system on $G_{\text{reg}}^{\iota\theta}$. Thus we have

$$\begin{aligned} (1.13.1) \quad \psi_* \bar{\mathbf{Q}}_l &\simeq \eta_* \xi_* \bar{\mathbf{Q}}_l \\ &\simeq H^\bullet(\mathbf{P}_1^n) \otimes \eta_* \bar{\mathbf{Q}}_l \\ &\simeq \bigoplus_{J \subseteq [1, n]} \bigoplus_{\lambda \in \mathcal{P}_n} V_\lambda \otimes \mathcal{L}_\lambda[-2|J|]. \end{aligned}$$

We have the following result (cf. [H1, Proposition 6.1, Proposition 6.2]).

Proposition 1.14. $\pi_* \bar{\mathbf{Q}}_l[\dim G^{\iota\theta}]$ and $(\pi_1)_* \bar{\mathbf{Q}}_l[\dim G_{\text{uni}}^{\iota\theta}]$ are semisimple complexes with S_n -action. They are decomposed as

$$(1.14.1) \quad \pi_* \bar{\mathbf{Q}}_l[\dim G^{\iota\theta}] \simeq H^\bullet(\mathbf{P}_1^n) \otimes \bigoplus_{\lambda \in \mathcal{P}_n} V_\lambda \otimes \text{IC}(G^{\iota\theta}, \mathcal{L}_\lambda)[\dim G^{\iota\theta}],$$

$$(1.14.2) \quad (\pi_1)_* \bar{\mathbf{Q}}_l[\dim G_{\text{uni}}^{\iota\theta}] \simeq H^\bullet(\mathbf{P}_1^n) \otimes \bigoplus_{\lambda \in \mathcal{P}_n} V_\lambda \otimes \text{IC}(\bar{\mathcal{O}}_{\lambda^\bullet}, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_{\lambda^\bullet}],$$

where $\mathcal{O}_{\lambda^\bullet}$ is a certain H -orbit in $G_{\text{uni}}^{\iota\theta}$, and the map $\lambda \mapsto \mathcal{O}_{\lambda^\bullet}$ gives a bijective correspondence between \mathcal{P}_n and $G_{\text{uni}}^{\iota\theta} / \sim_H$.

Proof. Put $\mathfrak{b} = \text{Lie } B$, $\mathfrak{t} = \text{Lie } T$ and $\mathfrak{u} = \text{Lie } U$. Then θ acts on \mathfrak{b} and \mathfrak{u} , and one can define $\mathfrak{b}^{-\theta}$, $\mathfrak{t}^{-\theta}$ and $\mathfrak{u}^{-\theta}$ as in the case of \mathfrak{g} . Let $\mathfrak{t}_{\text{reg}}^{-\theta}$ be the set of $t \in \mathfrak{t}^{-\theta}$ such

that all the eigenspaces of t have dimension 2, and put $\mathfrak{g}_{\text{reg}}^{-\theta} = \bigcup_{g \in H} g(\mathfrak{t}^{-\theta})g^{-1}$, the set of regular semisimple elements in $\mathfrak{g}^{-\theta}$. Put

$$\begin{aligned}\tilde{\mathfrak{g}}^{-\theta} &= \{(x, gB^\theta) \in \mathfrak{g}^{-\theta} \times H/B^\theta \mid \text{Ad}(g)^{-1}x \in \mathfrak{b}^{-\theta}\}, \\ \tilde{\mathfrak{g}}_{\text{nil}}^{-\theta} &= \{(x, gB^\theta) \in \mathfrak{g}_{\text{nil}}^{-\theta} \times H/B^\theta \mid \text{Ad}(g)^{-1}x \in \mathfrak{u}^{-\theta}\},\end{aligned}$$

and define maps $\pi' : \tilde{\mathfrak{g}}^{-\theta} \rightarrow \mathfrak{g}^{-\theta}$, $\pi'_1 : \tilde{\mathfrak{g}}_{\text{nil}}^{-\theta} \rightarrow \mathfrak{g}_{\text{nil}}^{-\theta}$ by the second projections. Let $\psi' : \pi'^{-1}(\mathfrak{g}_{\text{reg}}^{-\theta}) \rightarrow \mathfrak{g}_{\text{reg}}^{-\theta}$ be the restriction of π' on $\pi'^{-1}(\mathfrak{g}_{\text{reg}}^{-\theta})$. Then it is known by Henderson [H1, Proposition 6.1, Proposition 6.2], that $\psi'_* \bar{\mathbf{Q}}_l$, $\pi'_* \bar{\mathbf{Q}}_l$ and $(\pi'_1)_* \bar{\mathbf{Q}}_l$ are described as in (1.13.1), (1.14.1) and (1.14.2) by replacing irreducible local systems \mathcal{L}_λ on $G^{\iota\theta}$ by irreducible local systems \mathcal{L}'_λ on $\mathfrak{g}^{-\theta}$, and by replacing H -orbits in $G_{\text{uni}}^{\iota\theta}$ by H -orbits in $\mathfrak{g}_{\text{nil}}^{-\theta}$. Note that $\dim G^{\iota\theta} = \dim \mathfrak{g}^{-\theta}$ and $\dim G_{\text{uni}}^{\iota\theta} = \dim \mathfrak{g}_{\text{nil}}^{-\theta}$. Let $\log : G^{\iota\theta} \rightarrow \mathfrak{g}^{-\theta}$ be the map defined in 1.7. Then we have the following cartesian diagram

$$\begin{array}{ccc} \tilde{G}^{\iota\theta} & \xrightarrow{\pi} & G^{\iota\theta} \\ \widetilde{\log} \downarrow & & \downarrow \log \\ \tilde{\mathfrak{g}}^{-\theta} & \xrightarrow{\pi'} & \mathfrak{g}^{-\theta}, \end{array}$$

where $\widetilde{\log}(gB^\theta, x) = (gB^\theta, \log x)$. It follows from this that $(\log)^*(\pi'_* \bar{\mathbf{Q}}_l) \simeq \pi_* \bar{\mathbf{Q}}_l$. Moreover, one can check that this diagram induces an isomorphism

$$\tilde{G}_{\text{reg}}^{\iota\theta} \simeq \tilde{\mathfrak{g}}_{\text{reg}}^{-\theta} \times_{\mathfrak{g}_{\text{reg}}^{-\theta}} G_{\text{reg}}^{\iota\theta}.$$

It follows that $\log^*(\psi'_* \bar{\mathbf{Q}}_l) \simeq \psi_* \bar{\mathbf{Q}}_l$, and in particular, we have $\log^*(\text{IC}(\mathfrak{g}^{-\theta}, \mathcal{L}'_\lambda)) \simeq \text{IC}(G^{\iota\theta}, \mathcal{L}_\lambda)$ for each $\lambda \in \mathcal{P}_n$. Since π is a proper map, $\pi_* \bar{\mathbf{Q}}_l$ is a semisimple complex on $G^{\iota\theta}$. Then (1.14.1) follows from the corresponding result of Henderson. (1.14.2) also follows from the result of Henderson since $\widetilde{\log}$ gives an isomorphism between $\tilde{G}_{\text{uni}}^{\iota\theta}$ and $\tilde{\mathfrak{g}}_{\text{nil}}^{-\theta}$. \square

1.15. We consider the diagram

$$T^{\iota\theta} \xleftarrow{\alpha_0} \pi^{-1}(G_{\text{reg}}^{\iota\theta}) \xrightarrow{\psi} G_{\text{reg}}^{\iota\theta},$$

where α_0 is a map defined by $\alpha_0(x, gB^\theta) = p(g^{-1}xg)$ (here $p : B^\theta \rightarrow T^{\iota\theta}$ is the natural projection). Take a tame local system \mathcal{E} on $T^{\iota\theta}$ (i.e, a local system \mathcal{E} on $T^{\iota\theta}$ such that $\mathcal{E}^{\otimes m} \simeq \bar{\mathbf{Q}}_l$ for an integer m not divisible by $p = \text{ch } \mathbf{k}$), and consider the complex $\psi_* \alpha_0^* \mathcal{E}$ on $G_{\text{reg}}^{\iota\theta}$. Now the map ψ is decomposed $\psi = \eta \circ \xi$ as in (1.8.1). Accordingly, α_0 is also decomposed as $\alpha_0 = \alpha_1 \circ \xi$, where $\alpha_1 : H \times^{Z_H(T^{\iota\theta})} T_{\text{reg}}^{\iota\theta} \simeq H/Z_H(T^{\iota\theta}) \times T_{\text{reg}}^{\iota\theta} \rightarrow T^{\iota\theta}$ is a projection to the second factor. Since ξ is a \mathbf{P}_1^n -bundle, we see that $\xi_* \alpha_0^* \mathcal{E} \simeq H^\bullet(\mathbf{P}_1^n) \otimes \alpha_1^* \mathcal{E}$. Since η is a finite Galois covering with group S_n , we have

$$\eta_* \alpha_1^* \mathcal{E} \simeq \bigoplus_{\rho \in \mathcal{A}_{\mathcal{E}}^\wedge} \rho \otimes \mathcal{L}_\rho,$$

where $\mathcal{A}_\mathcal{E} = \text{End}(\eta_*\alpha_1^*\mathcal{E})$ and \mathcal{L}_ρ is an irreducible local system on $G_{\text{reg}}^{\iota\theta}$ defined by $\mathcal{L}_\rho = \text{End}_{\mathcal{A}_\mathcal{E}}(\rho, \eta_*\alpha_1^*\mathcal{E})$, corresponding to the irreducible module $\rho \in \mathcal{A}_\mathcal{E}^\wedge$. Note that $\mathcal{A}_\mathcal{E}$ is a twisted group algebra of the stabilizer $W_\mathcal{E}$ of \mathcal{E} in S_n . It follows that

$$(1.15.1) \quad \psi_*\alpha_0^*\mathcal{E} \simeq \bigoplus_{\rho \in \mathcal{A}_\mathcal{E}^\wedge} H^\bullet(\mathbf{P}_1^n) \otimes \rho \otimes \mathcal{L}_\rho.$$

We define a complex $K_{T,\mathcal{E}}$ on $G^{\iota\theta}$ by

$$(1.15.2) \quad K_{T,\mathcal{E}} = H^\bullet(\mathbf{P}_1^n) \otimes \bigoplus_{\rho \in \mathcal{A}_\mathcal{E}^\wedge} \rho \otimes \text{IC}(G^{\iota\theta}, \mathcal{L}_\rho)[\dim G^{\iota\theta}].$$

On the other hand, we consider a diagram

$$T^{\iota\theta} \xleftarrow{\alpha} \tilde{G}^{\iota\theta} \xrightarrow{\pi} G^{\iota\theta},$$

where $\alpha : \tilde{G}^{\iota\theta} \rightarrow T^{\iota\theta}$ is given by $\alpha(x, gB^\theta) = p(g^{-1}xg)$. We consider the complex $\pi_*\alpha^*\mathcal{E}[\dim G^{\iota\theta}]$ on $G^{\iota\theta}$. The following result is due to Grojonowski [Gr, Corollary 7.4] (see also [H2, Theorem 7.2]).

Theorem 1.16. $\pi_*\alpha^*\mathcal{E}[\dim G^{\iota\theta}] \simeq K_{T,\mathcal{E}}$ as semisimple complexes on $G^{\iota\theta}$.

1.17. Let $\mathcal{B}^H = H/B^\theta$ be the variety of Borel subgroups in H . For $x \in G^{\iota\theta}$, put $\mathcal{B}_x^H = \{gB^\theta \in \mathcal{B}^H \mid g^{-1}xg \in B^{\iota\theta}\}$. In the case where $x \in G_{\text{uni}}^{\iota\theta}$, $\mathcal{B}_x^H \simeq \pi_1^{-1}(x)$. We shall describe the structure of \mathcal{B}_x^H . First consider \mathcal{B}_s^H for a semisimple element $s \in G^{\iota\theta}$. By Remark 1.3 (iii), there exists $s' \in T^{\iota\theta}$ such that s' is H -conjugate to s . Let $W_H = N_H(T^\theta)/T^\theta$ be a Weyl group of H . For $s' \in T^{\iota\theta}$, let $W_{H,s'} = \{w \in W_H \mid w(s') = s'\}$, which is a Weyl group of $Z_H(s')$. Put

$$\mathcal{M}_s = \{g \in H \mid g^{-1}sg \in B^{\iota\theta}\}.$$

Then $Z_H(s) \times B^\theta$ acts on \mathcal{M}_s from left and right, and we consider the set of double cosets $Z_H(s) \backslash \mathcal{M}_s / B^\theta$. We note that this set is labelled by the set $\Gamma = W_{H,s'} \backslash W_H$. Clearly it is enough to check this for the case where $s = s' \in T^{\iota\theta}$. Then the claim follows from the following property.

(1.17.1) Let $s \in T^{\iota\theta}$. Assume that $g^{-1}sg \in B^{\iota\theta}$ for $g \in H$. Then there exists $g_1 \in Z_H(s)$ and $w \in W_H$ such that $gB^\theta = g_1wB^\theta$.

We show (1.17.1). Since ${}^gB \cap Z_G(s)$ is a θ -stable Borel subgroup in $Z_G(s)$, there exists a θ -stable maximal torus T' of $Z_G(s)$ contained in gB . Then by Remark 1.3 (ii), there exists $g_1 \in Z_H(s)$ such that ${}^{g_1}(B \cap Z_G(s)) = {}^gB \cap Z_G(s)$ and that ${}^{g_1}T = T'$. Then ${}^{g_1^{-1}}{}^gB$ is a Borel subgroup of G containing T , and so ${}^{g_1^{-1}}{}^gB^\theta$ is a Borel subgroup of H containing T^θ . Hence there exists $w \in W_H$ such that $gB^\theta = g_1wB^\theta \in \mathcal{B}^H$ as asserted.

Now we have

$$\begin{aligned}
\mathcal{B}_s^H &= \{gB^\theta \in \mathcal{B}^H \mid g^{-1}sg \in B^{\iota\theta}\} \\
&= \coprod_{\gamma \in \Gamma} Z_H(s)x_\gamma B^\theta / B^\theta \\
&= \coprod_{\gamma \in \Gamma} Z_H(s) / (Z_H(s) \cap x_\gamma B^\theta x_\gamma^{-1}) \\
&= \coprod_{\gamma \in \Gamma} Z_H(s) / B_\gamma^\theta,
\end{aligned}$$

where $x_\gamma \in H$ is a representative of the double coset in \mathcal{M}_s corresponding to $\gamma \in \Gamma$. Moreover, $B_\gamma = Z_G(s) \cap x_\gamma B x_\gamma^{-1}$ is a θ -stable Borel subgroup of $Z_G(s)$, and B_γ^θ is a Borel subgroup of $Z_H(s)$.

Next we consider the general case \mathcal{B}_{su}^H , where s is semisimple and u is unipotent in $G^{\iota\theta}$. We have

$$\begin{aligned}
\mathcal{B}_{su}^H &= \{gB^\theta \in \mathcal{B}^H \mid g^{-1}sg \in B^{\iota\theta}, g^{-1}ug \in B^{\iota\theta}\} \\
&= \coprod_{\gamma \in \Gamma} \{gB^\theta \in Z_H(s)x_\gamma B^\theta / B^\theta \mid g^{-1}ug \in B^{\iota\theta}\}.
\end{aligned}$$

By writing $gB^\theta = g_1x_\gamma B^\theta$ with $g_1 \in Z_H(s)$, the last formula turns out to be

$$\begin{aligned}
(1.17.2) \quad \mathcal{B}_{su}^H &= \coprod_{\gamma \in \Gamma} \{g_1 B_\gamma^\theta \in Z_H(s) / B_\gamma^\theta \mid g_1^{-1}ug_1 \in B_\gamma^{\iota\theta}\} \\
&= \coprod_{\gamma \in \Gamma} \mathcal{B}_u^{Z_H(s)},
\end{aligned}$$

where $\mathcal{B}^{Z_H(s)} \simeq Z_H(s) / B_\gamma^\theta$ is the variety of Borel subgroups of $Z_H(s)$.

It follows from the above computation, we have

$$(1.17.3) \quad \dim \mathcal{B}_{su}^H = \dim \mathcal{B}_u^{Z_H(s)}.$$

Thus we have the following generalization of Lemma 1.11 (i).

Lemma 1.18. *For $x \in G^{\iota\theta}$, we have $\dim \mathcal{B}_x^H = (\dim Z_H(x) - \text{rank } H) / 2$.*

2. H -ORBITS ON $G_{\text{uni}}^{\iota\theta} \times V$

2.1. We consider the action of H on the variety $G_{\text{uni}}^{\iota\theta} \times V$. By (1.7.2), the set of H -orbits in $G_{\text{uni}}^{\iota\theta} \times V$ is parametrized by the set $\mathcal{P}_{n,2}$. We denote by $\mathcal{O}_\lambda^H = \mathcal{O}_\lambda$ the H -orbit corresponding to $\lambda \in \mathcal{P}_{n,2}$ (the explicit correspondence is described below). For a given $(x, v) \in G_{\text{uni}}^{\iota\theta} \times V$, we say that (x, v) is of type λ if $(x, v) \in \mathcal{O}_\lambda$. Let M_n be the subspace of V spanned by e_1, \dots, e_n . Then A acts on M_n naturally, and the

set of A -orbits in $A_{\text{uni}} \times M_n$ is parametrized also by $\mathcal{P}_{n,2}$. Let \mathcal{O}_{λ}^A be the A -orbit in $A_{\text{uni}} \times M_n$ corresponding to $\lambda \in \mathcal{P}_{n,2}$. The correspondence $\mathcal{O}_{\lambda}^H \leftrightarrow \mathcal{O}_{\lambda}^A$ is given as follows ([AH, Theorem 6.1]); Take $(y, v) \in A_{\text{uni}} \times M_n$ such that $(y, v) \in \mathcal{O}_{\lambda}^A$. Then $x = y\theta(y)^{-1} \in G_{\text{uni}}^{\theta}$ and the H -orbit of (x, v) coincides with \mathcal{O}_{λ}^H .

On the other hand, for a given $(y, v) \in A_{\text{uni}} \times M_n$, the corresponding type is determined by the following procedure. Put $E_A^y = \{z \in \text{End}(M_n) \mid zy = yz\}$. Then $W = E_A^y v$ is an y -stable subspace of M_n . Let $\lambda^{(1)}$ be the Jordan type of $y|_W$, and $\lambda^{(2)}$ the Jordan type of $y|_{M_n/W}$. Then $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ gives the type of (y, v) . In particular, $\dim E_A^y v = \dim W = |\lambda^{(1)}|$. Let $E_H^x = \text{Lie } Z_H(x) \subset \mathfrak{g} = \text{End}(V)$. Then $E_H^x v$ is an x -stable subspace of V . On the other hand, (x, v) is regarded as an element in $G_{\text{uni}} \times V$. Then it is easy to see that the G -orbit of (x, v) is of type $\lambda \cup \lambda = (\lambda^{(1)} \cup \lambda^{(1)}, \lambda^{(2)} \cup \lambda^{(2)}) \in \mathcal{P}_{2n,2}$. We define $E_G^x = \{z \in \text{End}(V) \mid zx = xz\}$, and put $\widetilde{W} = E_G^x v$. Then \widetilde{W} is a subspace of V containing $E_H^x v$ with $\dim \widetilde{W} = 2|\lambda^{(1)}|$. We have the following lemmas.

Lemma 2.2. \widetilde{W} coincides with $E_H^x v$. Hence $\dim E_H^x v = 2|\lambda^{(1)}|$.

Proof. Let L be as in 1.4. Then $L^{\theta} = \{a\theta(a) \mid a \in A\}$ is a subgroup of H isomorphic to GL_n , and $E_{L^{\theta}}^x = \text{Lie } Z_{L^{\theta}}(x) \subset E_H^x$. Let $V = M_n \oplus M'_n$, where M'_n is the subspace of V generated by f_1, \dots, f_n . One can find $v' \in M'_n$ such that the type of (y', v') is λ , where $y' = \theta(y)^{-1} \in \theta(A)$. Then there exists $g \in \text{Lie } Z_H(x)$ such that $gv = v'$. We have $E_{L^{\theta}}^x v = E_A^y v$ and $E_{L^{\theta}}^x v' = E_{\theta(A)}^{y'} v'$. It follows that $E_H^x v$ contains a subspace $E_A^y v \oplus E_{\theta(A)}^{y'} v'$ whose dimension is equal to $2|\lambda^{(1)}|$. Hence $\widetilde{W} = E_H^x v$, and the lemma follows. \square

Lemma 2.3. Let $\mathcal{O}_H(x, v)$ be the H -orbit of $(x, v) \in G_{\text{uni}}^{\theta} \times V$, where (x, v) is of type λ , and $\mathcal{O}_H(x)$ the H -orbit of $x \in G_{\text{uni}}^{\theta}$. For $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ put $n(\lambda) = n(\lambda^{(1)} + \lambda^{(2)})$.

- (i) $\dim \mathcal{O}_H(x, v) = \dim \mathcal{O}_H(x) + 2|\lambda^{(1)}|$. In particular for $\mathcal{O}_{\lambda} = \mathcal{O}_H(x, v)$ we have

$$(2.3.1) \quad \dim \mathcal{O}_{\lambda} = 2n^2 - 2n - 4n(\lambda) + 2|\lambda^{(1)}|.$$

- (ii) $Z_H(x, v)$ is connected for any $(x, v) \in G_{\text{uni}}^{\theta} \times V$.

Proof. We consider the linear map $\varphi : E_H^x \rightarrow E_H^x v$ given by $g \mapsto gv$. Then by Lemma 2.2, $\dim E_H^x - \dim \ker \varphi = 2|\lambda^{(1)}|$. We note that

$$(2.3.2) \quad \dim \ker \varphi = \dim Z_H(x, v).$$

We show (2.3.2). Let H^+ be the set of elements of H such that -1 is not contained in its eigenvalues, and \mathfrak{h}^+ be the set of elements in $\mathfrak{h} = \text{Lie } H$ such that 1 is not contained in its eigenvalues. Then H^+ (resp. \mathfrak{h}^+) is open dense in H (resp. \mathfrak{h}). By the Cayley transform, $f : Z \mapsto (I + Z)(I - Z)^{-1}$ gives a bijection from \mathfrak{h}^+ to H^+ . The inverse f^{-1} is given by $z \mapsto (I + z)^{-1}(I - z)$. Then the map f induces a bijection from $\ker \varphi \cap \mathfrak{h}^+$ onto $Z_H(x, v) \cap H^+$. Since $\ker \varphi \cap \mathfrak{h}^+$ is open dense in $\ker \varphi$, and $Z_H(x, v) \cap H^+$ is open dense in $Z_H(x, v)$, we obtain (2.3.2).

By (2.3.2) and by $\dim E_H^x = \dim Z_H(x)$, we have $\dim Z_H(x, v) = \dim Z_H(x) - 2|\lambda^{(1)}|$. It follows that

$$\begin{aligned} \dim \mathcal{O}_H(x, v) &= \dim H - \dim Z_H(x, v) \\ &= \dim H - \dim Z_H(x) + 2|\lambda^{(1)}| \\ &= \dim \mathcal{O}_H(x) + 2|\lambda^{(1)}|. \end{aligned}$$

Now (2.3.1) follows from Lemma 1.11 (iii) since the Jordan type of x is $\lambda^{(1)} + \lambda^{(2)}$. This proves (i).

(ii) follows from the proof of (i) as follows; since $\ker \varphi \cap \mathfrak{h}^+$ is connected, $Z_H(x, v) \cap H^+$ is also connected. Since $Z_H(x, v) \cap H^+$ is open dense in $Z_H(x, v)$, we see that $Z_H(x, v)$ is connected. \square

2.4. We fix an isotropic flag $(0 = M_0 \subset M_1 \subset \cdots \subset M_n)$ whose stabilizer in H coincides with B^θ , where M_i is the subspace of V spanned by e_1, \dots, e_i . By fixing an integer m such that $0 \leq m \leq n$, we define

$$\begin{aligned} \tilde{\mathcal{X}}_{m, \text{uni}} &= \{(x, v, gB^\theta) \in G_{\text{uni}}^\theta \times V \times H/B^\theta \mid g^{-1}xg \in U^\theta, g^{-1}v \in M_m\}, \\ \mathcal{X}_{m, \text{uni}} &= \bigcup_{g \in H} g(U^\theta \times M_m). \end{aligned}$$

and a map $\pi_1^{(m)} : \tilde{\mathcal{X}}_{m, \text{uni}} \rightarrow G_{\text{uni}}^\theta \times V$ by $\pi_1^{(m)}(x, v, B^\theta) = (x, v)$. Clearly $\text{Im } \pi_1^{(m)} = \mathcal{X}_{m, \text{uni}}$. In the special case where $m = n$, we simply write $\tilde{\mathcal{X}}_{n, \text{uni}}, \mathcal{X}_{n, \text{uni}}, \pi_1^{(m)}$ as $\tilde{\mathcal{X}}_{\text{uni}}, \mathcal{X}_{\text{uni}}, \pi_1$. $\tilde{\mathcal{X}}_{m, \text{uni}}$ is smooth and irreducible since $\tilde{\mathcal{X}}_{m, \text{uni}} \simeq H \times^{B^\theta} (U^\theta \times M_m)$. Moreover, we have

$$(2.4.1) \quad \dim \tilde{\mathcal{X}}_{m, \text{uni}} = \dim H/B^\theta + \dim U^\theta + m = 2n^2 - n + m.$$

In order to obtain the dimension estimate for $\pi_1^{-1}(x, v)$, we consider the Steinberg variety defined as follows:

$$\begin{aligned} \mathcal{Z} &= \{(x, v, gB^\theta, g'B^\theta) \in G_{\text{uni}}^\theta \times V \times H/B^\theta \times H/B^\theta \\ &\quad \mid (x, v, gB^\theta) \in \mathcal{X}_{\text{uni}}, (x, v, g'B^\theta) \in \mathcal{X}_{\text{uni}}\}. \end{aligned}$$

Let $\nu_H = \dim U^\theta = n^2$. Let $W_n = N_H(T^\theta)/T^\theta$ be a Weyl group of $H \simeq Sp_{2n}$, the Weyl group of type C_n . We have the following lemma.

Lemma 2.5. (i) *The irreducible components of \mathcal{Z} are parametrized by $w \in W_n$, and have dimension $2\nu_H$.*
(ii) *Let $\pi_1 : \tilde{\mathcal{X}}_{\text{uni}} \rightarrow G_{\text{uni}}^\theta \times V$. For any $(x, v) \in \mathcal{O}_\mu$, we have $\dim \pi_1^{-1}(x, v) \leq \nu_H - \dim \mathcal{O}_\mu/2$.*

- (iii) Let c_μ be the number of irreducible components of $\pi_1^{-1}(x, v)$ for $(x, v) \in \mathcal{O}_\mu$ whose dimension is equal to $\nu_H - \dim \mathcal{O}_\mu/2$. Then we have

$$\sum_{\mu \in \mathcal{P}_{n,2}} c_\mu^2 \leq |W_n|.$$

Proof. Let $p : \mathcal{Z} \rightarrow H/B^\theta \times H/B^\theta$ be the projection to the last two factors. For each $w \in W_n$, let X_w be the H -orbit of (B^θ, wB^θ) in $H/B^\theta \times H/B^\theta$. We have $H/B^\theta \times H/B^\theta = \coprod_{w \in W_n} X_w$. Put $Z_w = p^{-1}(X_w)$. Then Z_w is a vector bundle over $X_w \simeq H/B^\theta \cap wB^\theta w^{-1}$ with fibre isomorphic to $(U^{\iota\theta} \cap wU^{\iota\theta} w^{-1}) \times (M_n \cap w(M_n))$. Let b_w be the number of i such that $w^{-1}(e_i) \in M_n$. Then $\dim(M_n \cap w(M_n)) = b_w$. By counting the root vectors in $U^{\iota\theta}$, we see that $\dim(U^{\iota\theta} \cap wU^{\iota\theta} w^{-1}) = \dim(U^{\iota\theta} \cap wU^{\iota\theta} w^{-1}) - b_w$. It follows that

$$\dim Z_w = \dim H - \dim T^\theta = 2\nu_H.$$

This implies that the set $\{\overline{Z}_w \mid w \in W_n\}$ gives rise to the set of irreducible components of \mathcal{Z} , any of their dimension is equal to $2\nu_H$. Hence (i) follows.

Let $f : \mathcal{Z} \rightarrow G_{\text{uni}}^{\iota\theta} \times V$ be the projection to the first two factors. For each H -orbit \mathcal{O} of $G_{\text{uni}}^{\iota\theta} \times V$ containing (x, v) , the fibre $f^{-1}(\mathcal{O})$ is isomorphic to a locally trivial fibration over \mathcal{O} with fibre isomorphic to $\pi_1^{-1}(x, v) \times \pi_1^{-1}(x, v)$, i.e., we have

$$f^{-1}(\mathcal{O}) \simeq H \times^{Z_H(x,v)} (\pi_1^{-1}(x, v) \times \pi_1^{-1}(x, v)).$$

It follows that

$$\dim f^{-1}(\mathcal{O}) = 2 \dim \pi_1^{-1}(x, v) + \dim \mathcal{O} \leq 2\nu_H.$$

(ii) follows from this.

We have $\mathcal{Z} = \coprod_{\mu \in \mathcal{P}_{n,2}} f^{-1}(\mathcal{O}_\mu)$. Take $(x, v) \in \mathcal{O}_\mu$ and let I_μ be the set of irreducible components of dimension $\nu_H - \dim \mathcal{O}_\mu/2$ of $\pi_1^{-1}(x, v)$. $Z_H(x, v)$ acts on $\pi_1^{-1}(x, v)$, which stabilizes each irreducible component of $\pi_1^{-1}(x, v)$ since $Z_H(x, v)$ is connected. For $X, Y \in I_\mu$, $H \times^{Z_H(x,v)} (X \times Y)$ gives an irreducible subset of $f^{-1}(\mathcal{O}_\mu)$ with dimension $2\nu_H$. It follows that its closure gives an irreducible component of $f^{-1}(\mathcal{O}_\mu)$, and that the number of irreducible components of $f^{-1}(\mathcal{O}_\mu)$ of dimension $2\nu_H$ is bigger than or equal to $|I_\mu|^2 = c_\mu^2$. The inequality in (iii) follows from this. \square

More generally, we have the following dimension estimate for $(\pi_1^{(m)})^{-1}(x, v)$ for $(x, v) \in G_{\text{uni}}^{\iota\theta} \times V$.

Corollary 2.6. *Let $\lambda = ((m), (n - m)) \in \mathcal{P}_{n,2}$. Let $(x, v) \in \mathcal{O}_\mu$ and assume that $\mu \leq \lambda$ (see 1.7). Then*

$$(2.6.1) \quad \begin{aligned} \dim(\pi_1^{(m)})^{-1}(x, v) &\leq 2n(\mu) + n - |\mu^{(1)}| \\ &= (\dim \mathcal{O}_\lambda - \dim \mathcal{O}_\mu)/2 + (n - m). \end{aligned}$$

Proof. By Lemma 2.5 (ii) and Lemma 2.3 (i), we have

$$\begin{aligned} \dim(\pi_1^{(n)})^{-1}(x, v) &\leq \nu_H - \dim \mathcal{O}_\mu / 2 \\ &= 2n(\mu) + n - |\mu^{(1)}|, \end{aligned}$$

Clearly $(\pi_1^{(m)})^{-1}(x, v) \subseteq (\pi_1^{(n)})^{-1}(x, v)$ and so $\dim(\pi_1^{(m)})^{-1}(x, v) \leq \dim(\pi_1^{(n)})^{-1}(x, v)$. Then (2.6.1) follows from the above inequality by noting that $\dim \mathcal{O}_\lambda = 2n^2 - 2n + 2m$. \square

By making use of Proposition 2.6, we have the following result. The assertion (ii) is known by Kato [Ka1, Theorem 1.2].

Proposition 2.7. (i) *Let $\lambda = ((m), (n - m)) \in \mathcal{P}_{n,2}$. Then $\pi_1^{(m)}$ is a map from $\tilde{\mathcal{X}}_{m,\text{uni}}$ onto $\mathcal{X}_{m,\text{uni}} = \overline{\mathcal{O}}_\lambda$.*

(ii) *In the case where $m = n$, \mathcal{X}_{uni} coincides with $\overline{\mathcal{O}}_\lambda$ with $\lambda = ((n), -)$. The map $\pi_1 : \tilde{\mathcal{X}}_{\text{uni}} \rightarrow \mathcal{X}_{\text{uni}}$ gives a resolution of singularities.*

Proof. We show (i). By Corollary 2.6, for each $(x, v) \in \mathcal{O}_\lambda$, $\dim(\pi_1^{(m)})^{-1}(x, v) \leq n - m$. We may assume that $x = y\theta(y)^{-1}$ for $y \in A_{\text{uni}}$. Put

$$X = \{(x, v, gB^\theta) \in (\pi_1^{(m)})^{-1}(x, v) \mid gM_m = E_A^y v\}.$$

Put $V_m = E_A^y v$. Then $x|_{V_m} \in GL_m$ is of type (m) , and $x|_{V_m^\perp/V_m} \in GL_{2(n-m)}^\theta$ is of type $(n - m)$. It follows that $X \simeq \pi_1'^{-1}(x|_{V_m}) \times \pi_1''^{-1}(x|_{V_m^\perp/V_m})$, where π_1'' is the map defined as in 1.10 replacing n by $n - m$, and π_1' is a similar map with respect to GL_m . It follows, by Lemma 1.11 (iii) that X is irreducible with $\dim X = n - m$. Hence we see that $\dim(\pi_1^{(m)})^{-1}(x, v) = n - m$. Since $(x, v) \in \text{Im } \pi_1^{(m)}$, we have $\mathcal{O}_\lambda \subset \text{Im } \pi_1^{(m)}$. It follows that $\dim(\pi_1^{(m)})^{-1}(\mathcal{O}_\lambda) = \dim \mathcal{O}_\lambda + (n - m) = 2n^2 - n + m$. Hence by (2.4.1), the closure of $(\pi_1^{(m)})^{-1}(\mathcal{O}_\lambda)$ coincides with $\tilde{\mathcal{X}}_{m,\text{uni}}$. This implies that $\tilde{\mathcal{X}}_{m,\text{uni}} = (\pi_1^{(m)})^{-1}(\overline{\mathcal{O}}_\lambda)$, and so $\text{Im } \pi_1^{(m)} = \overline{\mathcal{O}}_\lambda$ as asserted.

Next we show (ii). By (i), $\mathcal{X}_{\text{uni}} = \overline{\mathcal{O}}_\lambda$. We have $\dim \tilde{\mathcal{X}}_{\text{uni}} = \mathcal{X}_{\text{uni}}$, and $\tilde{\mathcal{X}}_{\text{uni}}$ is smooth, π_1 is proper. Hence in order to show that π_1 is a resolution of singularities, it is enough to see that the restriction of π_1 gives an isomorphism $\pi_1^{-1}(\mathcal{O}_\lambda) \xrightarrow{\sim} \mathcal{O}_\lambda$. We know that $\dim \pi_1^{-1}(x, v) = 0$ for $(x, v) \in \mathcal{O}_\lambda$ by Lemma 2.5 (ii). We consider the map $f : \mathcal{Z} \rightarrow \mathcal{X}_{\text{uni}}$ in the proof of Lemma 2.5. Since $f^{-1}(\mathcal{O}_\lambda)$ is open dense in \mathcal{Z} , it is irreducible. It follows that for $(x, v) \in \mathcal{O}_\lambda$, $\pi_1^{-1}(x, v)$ consists of one point. Since $f^{-1}(\mathcal{O}_\lambda)$ is a principal bundle with fibre $\pi_1^{-1}(x, v) \times \pi_1^{-1}(x, v)$, we see that $\mathcal{O}_\lambda \simeq f^{-1}(\mathcal{O}_\lambda) \simeq \pi_1^{-1}(\mathcal{O}_\lambda)$. This proves (ii). \square

3. INTERSECTION COHOMOLOGY ON $G_{\text{reg}}^{\iota\theta} \times V$

3.1. We consider the variety $G^{\iota\theta} \times V$ on which H acts diagonally. We define an isotropic subspace M_i of V by $M_i = \langle e_1, \dots, e_i \rangle$ for $i = 0, 1, \dots, n$. Thus the stabilizer of the isotropic flag (M_i) in H coincides with B^θ . For $0 \leq m \leq n$, we define varieties

$$\begin{aligned}\tilde{\mathcal{X}}_m &= \{(x, v, gB^\theta) \in G^{\iota\theta} \times V \times H/B^\theta \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}v \in M_m\}, \\ \mathcal{X}_m &= \bigcup_{g \in H} g(B^{\iota\theta} \times M_m).\end{aligned}$$

We define a map $\pi^{(m)} : \tilde{\mathcal{X}}_m \rightarrow G^{\iota\theta} \times V$ by $\pi^{(m)}(x, v, gB^\theta) = (x, v)$. Clearly $\text{Im } \pi^{(m)} = \mathcal{X}_m$. In the special case where $m = n$, we write $\tilde{\mathcal{X}}_n, \mathcal{X}_n$ and $\pi^{(n)}$ by $\tilde{\mathcal{X}}, \mathcal{X}$ and π .

Since $\tilde{\mathcal{X}}_m \simeq H \times^{B^\theta} (B^{\iota\theta} \times M_m)$, $\tilde{\mathcal{X}}_m$ is smooth and irreducible, and $\pi^{(m)}$ is a proper map. Hence \mathcal{X}_m is a closed irreducible subset of $G^{\iota\theta} \times V$. The dimension of $\tilde{\mathcal{X}}_m$ is computed as follows;

$$\begin{aligned}(3.1.1) \quad \dim \tilde{\mathcal{X}}_m &= \dim H/B^\theta + \dim B^{\iota\theta} + m \\ &= \dim U^\theta + \dim B^{\iota\theta} + m \\ &= \dim U + \dim T^{\iota\theta} + m \\ &= 2n^2 + m.\end{aligned}$$

We also define varieties

$$\begin{aligned}\tilde{\mathcal{Y}}_m &= \{(x, v, gB^\theta) \in G_{\text{reg}}^{\iota\theta} \times V \times H/B^\theta \mid g^{-1}xg \in B_{\text{reg}}^{\iota\theta}, g^{-1}v \in M_m\}, \\ \mathcal{Y}_m &= \bigcup_{g \in H} g(B_{\text{reg}}^{\iota\theta} \times M_m),\end{aligned}$$

and a map $\psi^{(m)} : \tilde{\mathcal{Y}}_m \rightarrow G^{\iota\theta} \times V$ by $\psi^{(m)}(x, v, gB^\theta) = (x, v)$. Clearly $\text{Im } \psi^{(m)} = \mathcal{Y}_m$. As before, in the case where $m = n$, we write $\tilde{\mathcal{Y}}_n, \mathcal{Y}_n$ and $\psi^{(n)}$ as $\tilde{\mathcal{Y}}, \mathcal{Y}$ and ψ . As in 1.8, $\tilde{\mathcal{Y}}_m$ can be expressed in the following form.

$$\begin{aligned}(3.1.2) \quad \tilde{\mathcal{Y}}_m &\simeq H \times^{B^\theta} (B_{\text{reg}}^{\iota\theta} \times M_m) \\ &\simeq H \times^{B^\theta \cap Z_H(T^{\iota\theta})} (T_{\text{reg}}^{\iota\theta} \times M_m).\end{aligned}$$

3.2. For a vector $v = \sum_{i=1}^n a_i e_i$ of M_n , put $\text{supp}(v) = \{i \mid a_i \neq 0\}$. For a subset I of $[1, n] = \{1, \dots, n\}$, put $M_I = \{v \in M_n \mid \text{supp}(v) = I\}$. As in 1.8, $Z_H(T^{\iota\theta})$ is isomorphic to $SL_2 \times \dots \times SL_2$, and under this identification $B^\theta \cap Z_H(T^{\iota\theta})$ corresponds to $B_2 \times \dots \times B_2$. Note that the action of $B^\theta \cap Z_H(T^{\iota\theta})$ on M_n is given by the action of its T^θ part. Hence $T_{\text{reg}}^{\iota\theta} \times M_I$ is $B^\theta \cap Z_H(T^{\iota\theta})$ -stable. Under the expressing (3.1.2) for $\tilde{\mathcal{Y}}$, we define, for $I \subset [1, n]$, a subvert $\tilde{\mathcal{Y}}_I$ of $\tilde{\mathcal{Y}}$ by

$$\tilde{\mathcal{Y}}_I \simeq H \times^{B^\theta \cap Z_H(T^{\iota\theta})} (T_{\text{reg}}^{\iota\theta} \times M_I).$$

We define a map $\psi_I : \tilde{\mathcal{Y}}_I \rightarrow \mathcal{Y}$ by $(x, v, gB^\theta) \mapsto (x, v)$. Then $\text{Im } \psi_I = \bigcup_{g \in H} g(T_{\text{reg}}^{\iota\theta} \times M_I)$ coincides with $\mathcal{Y}_m^0 = \mathcal{Y}_m \setminus \mathcal{Y}_{m-1}$ for $m = |I|$, which depends only on m . For

$I \subset [1, n]$ we define a parabolic subgroup $Z_H(T^{\iota\theta})_I$ of $Z_H(T^{\iota\theta})$ by the condition that the i -th factor is B_2 if $i \in I$, and is SL_2 otherwise. Since $Z_H(T^{\iota\theta})_I$ stabilizes M_I , one can define

$$\widehat{\mathcal{Y}}_I = H \times^{Z_H(T^{\iota\theta})_I} (T_{\text{reg}}^{\iota\theta} \times M_I).$$

Then the map ψ_I factors through $\widehat{\mathcal{Y}}_I$,

$$(3.2.1) \quad \psi_I : \widetilde{\mathcal{Y}}_I \xrightarrow{\xi_I} \widehat{\mathcal{Y}}_I \xrightarrow{\eta_I} \mathcal{Y}_m^0,$$

for $|I| = m$, where under the expression in (3.1.2), the map ξ_I is the natural surjection, and the map η_I is given by $g * (t, v) \mapsto (gtg^{-1}, gv)$ ($g * (t, v)$ denotes the $Z_H(T^{\iota\theta})_I$ -orbits of $(g, (t, v)) \in H \times (T_{\text{reg}}^{\iota\theta} \times M_I)$). Then ξ_I is a locally trivial fibration with fibre isomorphic to

$$(3.2.2) \quad Z_H(T^{\iota\theta})_I / (B^\theta \cap Z_H(T^{\iota\theta})) \simeq (SL_2/B_2)^{I'} \simeq \mathbf{P}_1^{I'},$$

where I' is the complement of I in $[1, n]$, and $(SL_2/B_2)^{I'}$ denotes the direct product of SL_2/B_2 with respect to the factors corresponding to I' , and similarly for $\mathbf{P}_1^{I'}$. Thus $\mathbf{P}_1^{I'} \simeq \mathbf{P}_1^{n-|I|}$.

Let $S_I \simeq S_{|I|} \times S_{n-|I|}$ be the subgroup of S_n stabilizing the set I . Then $\mathcal{W}_I = N_H(Z_H(T^{\iota\theta})_I) / Z_H(T^{\iota\theta})_I$ is isomorphic to S_I . In the case where $I = [1, m]$, we put $S_I = S_{\mathbf{m}}$, and $\mathcal{W}_I = \mathcal{W}_{\mathbf{m}}$ for $\mathbf{m} = (m, n-m)$. Thus $\mathcal{W}_{\mathbf{m}} \simeq S_{\mathbf{m}} \simeq S_m \times S_{n-m}$ under the natural isomorphism $\mathcal{W} \simeq S_n$ in 1.8. For $I \subset [1, n]$, \mathcal{W}_I acts on $\widetilde{\mathcal{Y}}_I$ and $\widehat{\mathcal{Y}}_I$ since $T_{\text{reg}}^{\iota\theta} \times M_I$ is stable by $N_H(Z_H(T^{\iota\theta})_I)$. Now the map $\eta_I : \widehat{\mathcal{Y}}_I \rightarrow \mathcal{Y}_m^0$ can be identified with the finite Galois covering with group \mathcal{W}_I ,

$$(3.2.3) \quad \widehat{\mathcal{Y}}_I \rightarrow \widehat{\mathcal{Y}}_I / \mathcal{W}_I \simeq \mathcal{Y}_m^0.$$

We have the following lemma.

Lemma 3.3. *Let the notations be as before.*

- (i) \mathcal{Y}_m is open dense in \mathcal{X}_m , and $\widetilde{\mathcal{Y}}_m$ is open dense in $\widetilde{\mathcal{X}}_m$.
- (ii) $\dim \widetilde{\mathcal{X}}_m = \dim \widetilde{\mathcal{Y}}_m = 2n^2 + m$.
- (iii) $\dim \mathcal{X}_m = \dim \mathcal{Y}_m = (2n^2 + m) - (n - m)$.
- (iv) $\mathcal{Y} = \coprod_{0 \leq m \leq n} \mathcal{Y}_m^0$ gives a stratification of \mathcal{Y} by smooth strata \mathcal{Y}_m^0 , and the map $\psi : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ is semismall with respect to this stratification.

Proof. Since $\widetilde{\mathcal{Y}}_m \simeq H \times^{B^\theta} (B_{\text{reg}}^{\iota\theta} \times M_m)$, and $B_{\text{reg}}^{\iota\theta} \times M_m$ is open dense in $B^{\iota\theta} \times M_m$, $\widetilde{\mathcal{Y}}_m$ is open dense in $\widetilde{\mathcal{X}}_m$. Since $\psi^{(m)}$ is a closed map, and since $\widetilde{\mathcal{Y}}_m = (\pi^{(m)})^{-1}(\mathcal{Y}_m)$, we see that \mathcal{Y}_m is open dense in \mathcal{X}_m . So (i) holds. (ii) follows from (3.1.1). By using the decomposition $\psi_I = \eta_I \circ \xi_I$ for $I = [1, m]$, we see that $\dim \widetilde{\mathcal{Y}}_m = \dim \mathcal{Y}_m + (n - m)$. Hence (iii) follows. For (iv), \mathcal{Y}_{m-1} is closed in \mathcal{Y}_m , and \mathcal{Y}_m^0 is an open dense smooth subset of \mathcal{Y}_m by the description of 3.2. Hence it gives the required stratification. Since $\dim \psi^{-1}(x, v) = n - m$ for $(x, v) \in \mathcal{Y}_m^0$ by 3.2, we have $\dim \psi^{-1}(x, v) = (\dim \mathcal{Y} - \dim \mathcal{Y}_m^0)/2$ by (iii). Thus (iv) holds. \square

3.4. For $0 \leq m \leq n$, we define $\tilde{\mathcal{Y}}_m^+$ as $\psi^{-1}(\mathcal{Y}_m^0)$. Then by 3.2, we have $\tilde{\mathcal{Y}}_m^+ = \coprod_I \tilde{\mathcal{Y}}_I$, where I runs over the subsets of $[1, n]$ such that $|I| = m$. $\tilde{\mathcal{Y}}_I$ are smooth and irreducible, and they form the connected components of $\tilde{\mathcal{Y}}_m^+$. Since $\mathcal{Y} = \coprod_{0 \leq m \leq n} \mathcal{Y}_m^0$, we have $\tilde{\mathcal{Y}} = \coprod_{0 \leq m \leq n} \tilde{\mathcal{Y}}_m^+$. In the case where $I = [1, m]$, we denote M_I by M_m^0 , and denote $\tilde{\mathcal{Y}}_I$ by $\tilde{\mathcal{Y}}_m^0$. M_m^0 is an open dense subset of M_m , and $\tilde{\mathcal{Y}}_m^0$ is an open dense subset of $\tilde{\mathcal{Y}}_m$. By (3.1.2), \mathcal{W} acts on $\tilde{\mathcal{Y}}$, which leaves $\tilde{\mathcal{Y}}_m^+$ stable for any m . Then we have

$$(3.4.1) \quad \tilde{\mathcal{Y}}_m^+ = \coprod_{\substack{I \subset [1, n] \\ |I| = m}} \tilde{\mathcal{Y}}_I = \coprod_{w \in \mathcal{W}/\mathcal{W}_{\mathbf{m}}} w(\tilde{\mathcal{Y}}_m^0).$$

We denote by $\psi_m : \tilde{\mathcal{Y}}_m^+ \rightarrow \mathcal{Y}_m^0$ the restriction of ψ on $\tilde{\mathcal{Y}}_m^+$. Then ψ_m is \mathcal{W} -equivariant with respect to the trivial action of \mathcal{W} on \mathcal{Y}_m^0 .

We consider the diagram

$$T^{\iota\theta} \xleftarrow{\alpha_0} \tilde{\mathcal{Y}} \xrightarrow{\psi} \mathcal{Y},$$

where $\alpha_0 : \tilde{\mathcal{Y}} \rightarrow T^{\iota\theta}$ is given by $\alpha_0(x, v, gB^\theta) = p(g^{-1}xg)$. Let \mathcal{E} be a tame local system on $T^{\iota\theta}$, and we consider the complex $\psi_*\alpha_0^*\mathcal{E}$ on \mathcal{Y} . One can define a map $\alpha_I : \tilde{\mathcal{Y}}_I \rightarrow T^{\iota\theta}$ compatible with α_0 with respect to the inclusion $\tilde{\mathcal{Y}}_I \hookrightarrow \tilde{\mathcal{Y}}$. Then by (3.4.1), we have

$$(3.4.2) \quad (\psi_m)_*\alpha_0^*\mathcal{E}|_{\tilde{\mathcal{Y}}_m^+} \simeq \bigoplus_{\substack{I \subset [1, n] \\ |I| = m}} (\psi_I)_*\alpha_I^*\mathcal{E}.$$

We define a map $\beta_I : \hat{\mathcal{Y}}_I \rightarrow T^{\iota\theta}$ by $g * (t, v) \mapsto t$. Then $\alpha_I = \beta_I \circ \xi_I$. Let $\mathcal{E}_I = \beta_I^*\mathcal{E}$ be a local system on $\hat{\mathcal{Y}}_I$ and $\mathcal{W}_{\mathcal{E}_I}$ the stabilizer of \mathcal{E}_I in \mathcal{W}_I . In the case where $I = [1, m]$, we put $\mathcal{W}_{\mathcal{E}_I} = \mathcal{W}_{\mathbf{m}, \mathcal{E}}$. Also put $\mathcal{W}_{\mathcal{E}_I} = \mathcal{W}_{\mathcal{E}}$ for $I = [1, n]$. $\mathcal{W}_{\mathcal{E}}$ acts on $(\psi_m)_*\alpha_0^*\mathcal{E}|_{\tilde{\mathcal{Y}}_m^+}$ as automorphisms of complexes, and permutes each direct summand $(\psi_I)_*\alpha_I^*\mathcal{E}$ according to the permutation of the sets I by S_n . Put $\text{End}((\eta_I)_*\mathcal{E}_I) = \mathcal{A}_{\mathcal{E}_I}$. Since η_I is a finite Galois covering with group \mathcal{W}_I , $(\eta_I)_*\mathcal{E}_I$ is decomposed as

$$(3.4.3) \quad (\eta_I)_*\mathcal{E}_I \simeq \bigoplus_{\rho \in \mathcal{A}_{\mathcal{E}_I}^\wedge} \rho \otimes \mathcal{L}_\rho,$$

where $\mathcal{L}_\rho = \text{Hom}(\rho, (\eta_I)_*\mathcal{E}_I)$ is a simple local system on \mathcal{Y}_m^0 . We note that $\mathcal{A}_{\mathcal{E}_I}$ is canonically isomorphic to the group algebra $\bar{\mathbf{Q}}_l[\mathcal{W}_{\mathcal{E}_I}]$. In fact, by [L3, 10.2], $\mathcal{A}_{\mathcal{E}_I}$ is a twisted group algebra of $\mathcal{W}_{\mathcal{E}_I}$. However, in the case where \mathcal{E} is the constant sheaf $\bar{\mathbf{Q}}_l$, $\mathcal{A}_{\mathcal{E}_I}$ is canonically isomorphic to the group algebra $\bar{\mathbf{Q}}_l[\mathcal{W}_I]$. In the general case, there exists a canonical embedding $\mathcal{A}_{\mathcal{E}_I} \hookrightarrow \mathcal{A}_{(\bar{\mathbf{Q}}_l)_I} \simeq \bar{\mathbf{Q}}_l[\mathcal{W}_I]$, and this implies that $\mathcal{A}_{\mathcal{E}_I}$ is the group algebra (see [L4, 2.4]).

3.5. Since ψ_m is proper and $\tilde{\mathcal{Y}}_I$ is closed in $\tilde{\mathcal{Y}}_m^+$, ψ_I is proper. Hence ξ_I is also proper. Since ξ_I is a $\mathbf{P}_1^{I'}$ bundle, we have $(\xi_I)_*\alpha_I^*\mathcal{E} \simeq H^\bullet(\mathbf{P}_1^{I'}) \otimes \mathcal{E}_I$. It follows that

$$(3.5.1) \quad (\psi_I)_*\alpha_I^*\mathcal{E} \simeq (\eta_I)_*(\xi_I)_*\alpha_I^*\mathcal{E} \simeq H^\bullet(\mathbf{P}_1^{I'}) \otimes (\eta_I)_*\mathcal{E}_I.$$

Now $\mathbf{P}_1^{I'}$ is the flag variety of the group $(SL_2)^{I'}$ whose Weyl group is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{I'}$. Let T_2 be a maximal torus of B_2 . Then SL_2/T_2 is a vector bundle over SL_2/B_2 , and $(\mathbf{Z}/2\mathbf{Z})^{I'}$ acts on $H^\bullet(\mathbf{P}_1^{I'}, \bar{\mathbf{Q}}_l) \simeq H^\bullet((SL_2/T_2)^{I'}, \bar{\mathbf{Q}}_l)$ naturally (the Springer action of $(\mathbf{Z}/2\mathbf{Z})^{I'}$ on $(SL_2/B_2)^{I'}$). We put $\mathcal{A}_{\mathcal{E}_I} = \mathcal{A}_{\mathbf{m}, \mathcal{E}}$ if $I = [1, m]$, and put $\mathcal{A}_{\mathcal{E}_I} = \mathcal{A}_{\mathcal{E}}$ if $I = [1, n]$. Then $\mathcal{A}_{\mathbf{m}, \mathcal{E}}$ is a subalgebra of $\mathcal{A}_{\mathcal{E}}$. Recall that $W_n = S_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n$ is the Weyl group of type C_n , and put $\tilde{\mathcal{W}} = \mathcal{W} \ltimes (\mathbf{Z}/2\mathbf{Z})^n$. We define a subgroup $\tilde{\mathcal{W}}_{\mathcal{E}}$ (resp. $\tilde{\mathcal{W}}_{\mathcal{E}_I}$) of $\tilde{\mathcal{W}}$ by $\tilde{\mathcal{W}}_{\mathcal{E}} = \mathcal{W}_{\mathcal{E}} \ltimes (\mathbf{Z}/2\mathbf{Z})^n$ (resp. $\tilde{\mathcal{W}}_{\mathcal{E}_I} = \mathcal{W}_{\mathcal{E}_I} \ltimes (\mathbf{Z}/2\mathbf{Z})^n$). We define an algebra $\tilde{\mathcal{A}}_{\mathcal{E}}$ by $\tilde{\mathcal{A}}_{\mathcal{E}} = \bar{\mathbf{Q}}_l[\tilde{\mathcal{W}}_{\mathcal{E}}]$. Similarly we define $\tilde{\mathcal{A}}_{\mathcal{E}_I}$. It follows from the above discussion that $\tilde{\mathcal{A}}_{\mathcal{E}_I}$ acts on $(\psi_I)_*\alpha_I^*\mathcal{E}$. (The action of $\tilde{\mathcal{A}}_{\mathcal{E}_I}$ on $(\eta_I)_*\mathcal{E}_I$ is the trivial extension of the action of $\mathcal{A}_{\mathcal{E}_I}$, and that of $\tilde{\mathcal{A}}_{\mathcal{E}_I}$ on $H^\bullet(\mathbf{P}_1^{I'})$ is obtained from the action of $(\mathbf{Z}/2\mathbf{Z})^n$ together with $\mathcal{A}_{\mathcal{E}_I}$, where $(\mathbf{Z}/2\mathbf{Z})^{I'}$ acts as defined above, and $(\mathbf{Z}/2\mathbf{Z})^I$ acts trivially). We put $\tilde{\mathcal{A}}_{\mathbf{m}, \mathcal{E}} = \tilde{\mathcal{A}}_{\mathcal{E}_I}$ for $I = [1, m]$. Then in view of (3.4.2), (3.4.3) and (3.5.1), we see that

$$(3.5.2) \quad (\psi_m)_*\alpha_0^*\mathcal{E}|_{\tilde{\mathcal{Y}}_m^+} \simeq \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}^\wedge} \text{Ind}_{\tilde{\mathcal{A}}_{\mathbf{m}, \mathcal{E}}}^{\tilde{\mathcal{A}}_{\mathcal{E}}} (H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho) \otimes \mathcal{L}_\rho,$$

where ρ is regarded as a $\tilde{\mathcal{A}}_{\mathbf{m}, \mathcal{E}}$ -module through the trivial extension from $\mathcal{A}_{\mathbf{m}, \mathcal{E}}$ to $\tilde{\mathcal{A}}_{\mathbf{m}, \mathcal{E}}$, and $H^\bullet(\mathbf{P}_1^{n-m})$ is regarded as a $S_{n-m} \ltimes (\mathbf{Z}/2\mathbf{Z})^{n-m}$ -module through the Springer action of $(\mathbf{Z}/2\mathbf{Z})^{n-m}$ and the S_{n-m} action arising from the permutations of factors \mathbf{P}_1 .

The group $\mathcal{W}_{\mathcal{E}_I}$ is decomposed as $\mathcal{W}_1 \times \mathcal{W}_2$, where \mathcal{W}_1 (resp. \mathcal{W}_2) is the stabilizer of \mathcal{E} in S_I (resp. $S_{I'}$) under the isomorphism $\mathcal{W}_I \simeq S_I \times S_{I'}$. Accordingly, the algebra $\mathcal{A}_{\mathbf{m}, \mathcal{E}}$ can be decomposed as $\mathcal{A}_{\mathbf{m}, \mathcal{E}} \simeq \mathcal{A}_1 \otimes \mathcal{A}_2$, where \mathcal{A}_1 (resp. \mathcal{A}_2) is the twisted group algebra of \mathcal{W}_1 (resp. \mathcal{W}_2) for $I = [1, m]$. Then an irreducible $\mathcal{A}_{\mathbf{m}, \mathcal{E}}$ -module ρ can be extended to an $\tilde{\mathcal{A}}_{\mathbf{m}, \mathcal{E}}$ -module ρ' , where $\mathbf{Z}/2\mathbf{Z}$ acts trivially on $\mathcal{A}_{\mathbf{m}, \mathcal{E}}$ if it belongs to I , and non-trivially if it belongs to I' . We denote by \tilde{V}_ρ the induced $\tilde{\mathcal{A}}_{\mathcal{E}}$ -module $\tilde{\mathcal{A}}_{\mathcal{E}} \otimes_{\tilde{\mathcal{A}}_{\mathbf{m}, \mathcal{E}}} \rho'$.

We show the following result.

Proposition 3.6. *$\psi_*\alpha_0^*\mathcal{E}[d_n]$ is a semisimple perverse sheaf on \mathcal{Y} , equipped with $\tilde{\mathcal{A}}_{\mathcal{E}}$ -action, and is decomposed as*

$$\psi_*\alpha_0^*\mathcal{E}[d_n] \simeq \bigoplus_{0 \leq m \leq n} \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}^\wedge} \tilde{V}_\rho \otimes \text{IC}(\mathcal{Y}_m, \mathcal{L}_\rho)[d_m],$$

where $d_m = \dim \mathcal{Y}_m$. Moreover, $\text{IC}(\mathcal{Y}_m, \mathcal{L}_\rho)$ is a constructible sheaf on \mathcal{Y}_m .

Proof. For each $m \leq n$, let $\bar{\psi}_m$ be the restriction of ψ on $\psi^{-1}(\mathcal{Y}_m)$, and $\tilde{\mathcal{E}}_m$ be the restriction of $\alpha_0^* \mathcal{E}$ on $\psi^{-1}(\mathcal{Y}_m)$. The following fact holds.

(3.6.1) For each m , $\mathrm{IC}(\mathcal{Y}_m, \mathcal{L}_\rho)$ is a constructible sheaf on \mathcal{Y}_m , and we have

$$\begin{aligned} (\bar{\psi}_m)_* \tilde{\mathcal{E}}_m[d_m] &\simeq \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \varepsilon}^\wedge} \mathrm{Ind}_{\tilde{\mathcal{A}}_{\mathbf{m}, \varepsilon}}^{\tilde{\mathcal{A}}_\varepsilon} (H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho) \otimes \mathrm{IC}(\mathcal{Y}_m, \mathcal{L}_\rho)[d_m] \\ &\quad \oplus \bigoplus_{0 \leq m' < m} \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}', \varepsilon}^\wedge} \tilde{V}_\rho \otimes \mathrm{IC}(\mathcal{Y}_{m'}, \mathcal{L}_\rho)[d_{m'} - 2(n-m)] \end{aligned}$$

where $\tilde{\mathcal{A}}_{\mathbf{m}, \varepsilon}$ -module $H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho$ is given as in (3.5.2), $\tilde{\mathcal{A}}_\varepsilon$ -module \tilde{V}_ρ is as in 3.5.

Note that (3.6.1) for $m = n$ implies the proposition. First we show that

(3.6.2) $\mathrm{IC}(\mathcal{Y}_m, \mathcal{L}_\rho)$ is a constructible sheaf on \mathcal{Y}_m .

Recall the map $\eta_I : \widehat{\mathcal{Y}}_m \rightarrow \mathcal{Y}_m^0$. We define a variety $\widehat{\mathcal{Y}}_m$ by

$$\widehat{\mathcal{Y}}_m = H \times^{Z_H(T^{u\theta})_I} (T_{\mathrm{reg}}^{u\theta} \times \overline{M}_I),$$

where \overline{M}_I is the closure of M_I in V . We define a morphism $\bar{\eta}_I : \widehat{\mathcal{Y}}_m \rightarrow \mathcal{Y}_m$ by $g * (t, v) \mapsto (gtg^{-1}, gv)$. Then $\bar{\eta}_I$ is a finite morphism and $\widehat{\mathcal{Y}}_m$ is smooth. Let $\bar{\beta}_I : \widehat{\mathcal{Y}}_m \rightarrow T^{u\theta}$ be the map defined by $g * (t, v) \mapsto t$, and put $\bar{\mathcal{E}}_I = \bar{\beta}_I^* \mathcal{E}$. Then $(\bar{\eta}_I)_* \bar{\beta}_I$ is a perfect sheaf in the sense of [L2, (5.4.4)]. In particular, $(\bar{\eta}_I)_* \bar{\mathcal{E}}_I$ is a direct sum of intersection cohomology complexes on \mathcal{Y}_m which are constructible sheaves. Since $(\bar{\eta}_I)_* \bar{\mathcal{E}}_I|_{\mathcal{Y}_m^0} = (\eta_I)_* \mathcal{E}_I$, we see that $\mathrm{IC}(\mathcal{Y}_m, \mathcal{L}_\rho)$ coincides with one of the simple direct summands in $(\bar{\eta}_I)_* \bar{\mathcal{E}}_I$. Hence (3.6.2) holds.

For each m , $\mathbf{m} = (m, n-m)$ and $0 \leq i \leq n-m$, define

$$\begin{aligned} \mathcal{L}_m &= \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \varepsilon}^\wedge} \tilde{V}_\rho \otimes \mathcal{L}_\rho, \\ \mathcal{L}_{m,i} &= \bigoplus_{\substack{J \subset [1, n-m] \\ |J|=i}} \mathcal{L}_m. \end{aligned}$$

Then the formula in (3.6.1) is equivalent to the following formula.

$$(3.6.3) \quad (\bar{\psi}_m)_* \tilde{\mathcal{E}}_m \simeq \bigoplus_{0 \leq i \leq n-m} \mathrm{IC}(\mathcal{Y}_m, \mathcal{L}_{m,i})[-2i] \oplus \bigoplus_{0 \leq m' < m} \mathrm{IC}(\mathcal{Y}_{m'}, \mathcal{L}_{m'})[2m' - 2n].$$

We show (3.6.3) by induction on m . In the case where $m = 0$, (3.6.3) follows from (1.13.1). Assume that (3.6.3) holds for m . Since $\bar{\psi}_{m+1}$ is a proper map, by the decomposition theorem $(\bar{\psi}_{m+1})_* \tilde{\mathcal{E}}_{m+1}$ is a direct sum of the complexes $A[i]$ for various simple perverse sheaf A on \mathcal{Y}_{m+1} . Suppose that $\mathrm{supp} A$ is not contained in \mathcal{Y}_m . Then $\mathrm{supp} A \cap \mathcal{Y}_{m+1}^0 \neq \emptyset$, and $A|_{\mathcal{Y}_{m+1}^0}$ appears as a direct summand in

the decomposition $(\psi_m)_*\alpha_0^*\mathcal{E}|_{\tilde{\mathcal{Y}}_m^+}$. It follows, by (3.5.2), that $A|_{\mathcal{Y}_{m+1}^0}$ coincides with \mathcal{L}_ρ for some $\rho \in \mathcal{A}_{\mathbf{m}+1, \mathcal{E}}$ with $\mathbf{m} + 1 = (m + 1, n - m - 1)$. Hence A coincides with $\mathrm{IC}(\mathcal{Y}_{m+1}, \mathcal{L}_\rho)[d_{m+1}]$. This implies that the direct sum of $A[i]$ appearing in $(\bar{\psi}_{m+1})_*\tilde{\mathcal{E}}_{m+1}$ such that $\mathrm{supp} A \cap \mathcal{Y}_{m+1}^0 \neq \emptyset$ is given by

$$(3.6.4) \quad \bigoplus_{0 \leq i \leq n-m-1} \mathrm{IC}(\mathcal{Y}_{m+1}, \mathcal{L}_{m+1,i})[-2i].$$

Next suppose that $\mathrm{supp} A \subset \mathcal{Y}_m$. Then $A[i]$ appears as a direct summand of the semisimple complex $(\bar{\psi}_m)_*\tilde{\mathcal{E}}_m[d_m]$. By induction hypothesis, the complex $(\bar{\psi}_m)_*\tilde{\mathcal{E}}_m$ is decomposed as in (3.6.3). Comparing (3.6.3) and (3.6.4), and by using (3.6.2), we see that each factor $\mathrm{IC}(\mathcal{Y}_{m'}, \mathcal{L}_{m'})[2m' - 2n]$ in $(\bar{\psi}_m)_*\tilde{\mathcal{E}}_m$ for $0 \leq m' \leq m$ appears in $(\bar{\psi}_{m+1})_*\tilde{\mathcal{E}}_{m+1}$ as a direct summand. Moreover, we see that $R^{2i}(\bar{\psi}_{m+1})_*\tilde{\mathcal{E}}_{m+1} = \mathrm{IC}(\mathcal{Y}_{m+1}, \mathcal{L}_{m+1,i})$ and $R^{2i}(\bar{\psi}_m)_*\tilde{\mathcal{E}}_m = \mathrm{IC}(\mathcal{Y}_m, \mathcal{L}_{m,i})$ for $0 \leq i < n - m$. Hence we have $\mathrm{IC}(\mathcal{Y}_{m+1}, \mathcal{L}_{m+1,i})|_{\mathcal{Y}_m} = \mathrm{IC}(\mathcal{Y}_m, \mathcal{L}_{m,i})$. It follows that each factor $\mathrm{IC}(\mathcal{Y}_m, \mathcal{L}_{m,i})[-2i]$ in (3.6.3) for $0 \leq i < n - m$ is absorbed to $\mathrm{IC}(\mathcal{Y}_{m+1}, \mathcal{L}_{m+1,i})[-2i]$ in (3.6.4). This shows that (3.6.3) holds also for $m + 1$. Hence (3.6.3) is proved and the proposition follows. \square

4. INTERSECTION COHOMOLOGY ON $G^{u\theta} \times V$

4.1. We keep the notation in Section 3. In view of Proposition 3.6, we define a semisimple perverse sheaf $K_{T, \mathcal{E}}$ on \mathcal{X} by

$$(4.1.1) \quad K_{T, \mathcal{E}} = \bigoplus_{0 \leq m \leq n} \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}} \tilde{V}_\rho \otimes \mathrm{IC}(\mathcal{X}_m, \mathcal{L}_\rho)[d_m],$$

which is the *DGM*-extension of $\psi_*\alpha_0^*\mathcal{E}[d_n]$. We call $\mathrm{IC}(\mathcal{X}_m, \mathcal{L}_\rho)[\dim \mathcal{X}_m]$ extended by 0 to \mathcal{X} , for various $\rho \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}$, character sheaves on $\mathcal{X} = G^{u\theta} \times V$. We consider a diagram

$$T^{u\theta} \xleftarrow{\alpha} \tilde{\mathcal{X}} \xrightarrow{\pi} \mathcal{X},$$

where $\alpha : \tilde{\mathcal{X}} \rightarrow T^{u\theta}$ is defined by $\alpha(x, v, gB^\theta) = p(g^{-1}xg)$. For a tame local system \mathcal{E} on $T^{u\theta}$, we consider a complex $\pi_*\alpha^*\mathcal{E}$ on \mathcal{X} . The following result is an analogy of Theorem 1.16.

Theorem 4.2. $\pi_*\alpha^*\mathcal{E}[\dim \mathcal{X}] \simeq K_{T, \mathcal{E}}$ as perverse sheaves on \mathcal{X} .

4.3. The theorem will be proved in 4.9 after some preliminaries. For each $m \leq n$, put $\mathcal{X}_m^0 = \mathcal{X}_m \setminus \mathcal{X}_{m-1}$. The set \mathcal{X}_m^0 is described as follows; First consider the unipotent part $\mathcal{X}_{m, \mathrm{uni}}^0$ of \mathcal{X}_m^0 . Since $\mathcal{X}_{m, \mathrm{uni}}$ is the closure of \mathcal{O}_λ with $\lambda = ((m), (n-m))$ (see Proposition 2.7), we see by using (1.7.3) that

(4.3.1) $\mathcal{X}_{m,\text{uni}}^0$ is the union of \mathcal{O}_μ such that $\mu = (\mu^{(1)}, \mu^{(2)})$ with $\mu_1^{(1)} = m$, for the partition $\mu^{(1)} : \mu_1^{(1)} \geq \mu_2^{(1)} \geq \dots$. In particular, (x, v) is contained in \mathcal{X}_m^0 if and only if $\dim \mathbf{k}[x]v = m$, where $\mathbf{k}[x]v$ denotes the subspace of V spanned by v, xv, x^2v, \dots .

More generally, consider $(x, v) \in G^{u\theta} \times V$. Let $x = su$ be the Jordan decomposition of $x \in G^{u\theta}$ and consider the decomposition $V = V_1 \oplus \dots \oplus V_t$ into eigenspaces of s . Then $Z_G(s) \simeq GL_{2n_1} \times \dots \times GL_{2n_t}$ with $\dim V_i = 2n_i$. Put $G_i = GL_{2n_i}$ for each i . Then $Z_G(s)$ is θ -stable, and θ stabilizes each factor so that $Z_H(s) \simeq G_1^\theta \times \dots \times G_t^\theta$ with $G_i^\theta \simeq Sp(V_i)$. Let $v = v_1 + \dots + v_t$ be the decomposition of $v \in V$ with $v_i \in V_i$. Let u_i be the restriction of u on V_i . Then $(u_i, v_i) \in G_i^{u_i\theta} \times V_i$. We denote by $\mathcal{X}_{m_i,\text{uni}}^{G_i,0}$ the subvariety of $G_{i,\text{uni}}^{u_i\theta} \times V_i$ defined in a similar way as $\mathcal{X}_{m,\text{uni}}^0$ for $G_{\text{uni}}^{u\theta} \times V$. Then we have

(4.3.2) (x, v) is contained in \mathcal{X}_m^0 if and only if $(u_i, v_i) \in \mathcal{X}_{m_i,\text{uni}}^{G_i,0}$ with $\sum_{i=1}^t m_i = m$.

For each $(x, v) \in G^{u\theta} \times V$, let (u_i, v_i) be defined as above. We define a subspace $W = W(x, v)$ of V by $W = \mathbf{k}[u_1]v_1 \oplus \dots \oplus \mathbf{k}[u_t]v_t$. Thus W is an isotropic, x -stable subspace of V containing v . Then (4.3.2) can be rewritten as

$$(4.3.3) \quad \mathcal{X}_m^0 = \{(x, v) \in G^{u\theta} \times V \mid \dim W(x, v) = m\}.$$

Recall the map $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$. For an integer $0 \leq m \leq n$, we define a locally closed subvariety $\tilde{\mathcal{X}}_m^+$ of $\tilde{\mathcal{X}}$ by $\tilde{\mathcal{X}}_m^+ = \pi^{-1}(\mathcal{X}_m^0)$. Then \mathcal{Y}_m^0 is open dense in \mathcal{X}_m^0 and $\tilde{\mathcal{Y}}_m^+$ is an open subset of $\tilde{\mathcal{X}}_m^+$. For $(x, v, gB^\theta) \in \tilde{\mathcal{X}}_m^+$, we define its level $I \subset [1, n]$ as follows; assume that $(x, v) \in B^{u\theta} \times M_n$ and that $x = su$, the Jordan decomposition of g with $s \in T^{u\theta}$. Then M_n is s -stable, and is decomposed as $M_n = M_{n_1}^{[1]} \oplus \dots \oplus M_{n_t}^{[t]}$, where $M_{n_i}^{[i]} = M_n \cap V_i$ is a maximal isotropic subspace of V_i . Note that $M_n = \langle e_1, \dots, e_n \rangle$. Since $\{e_i\}$ are weight vectors for T , $M_{n_i}^{[i]}$ has a basis e_{j_1}, e_{j_2}, \dots , with $j_1 < j_2 < \dots < j_{n_i}$. Let (u_i, v_i) be as above, and assume that $\dim \mathbf{k}[u_i]v_i = m_i$. Let us define a subset I_i of $\{j_1, \dots, j_{n_i}\}$ by choosing first m_i numbers. Hence $|I_i| = m_i$. We define $I = \coprod_i I_i$. Since $(x, v) \in \mathcal{X}_m^0$, we have $|I| = m$. Note that the attachment $(x, v) \mapsto I$ depends only on the B^θ -conjugate of (x, v) . Thus we have a well-defined map $(x, v, gB^\theta) \mapsto I$. We define a subvariety $\tilde{\mathcal{X}}_I$ of $\tilde{\mathcal{X}}_m^+$ by

$$\tilde{\mathcal{X}}_I = \{(x, v, gB^\theta) \in \tilde{\mathcal{X}}_m^+ \mid (x, v, gB^\theta) \mapsto I\}.$$

We show the following lemma.

Lemma 4.4. $\tilde{\mathcal{X}}_m^+$ is decomposed as

$$\tilde{\mathcal{X}}_m^+ = \coprod_{\substack{I \subset [1, n] \\ |I| = m}} \tilde{\mathcal{X}}_I,$$

where $\tilde{\mathcal{X}}_I$ is an irreducible component of $\tilde{\mathcal{X}}_m^+$ for each I .

Proof. It is clear from the definition that $\tilde{\mathcal{X}}_m^+$ is a disjoint union of various $\tilde{\mathcal{X}}_I$, and that $\tilde{\mathcal{X}}_I$ contains $\tilde{\mathcal{Y}}_I$. One can check that $\tilde{\mathcal{Y}}_I$ is open dense in $\tilde{\mathcal{X}}_I$. Since $\tilde{\mathcal{Y}}_m^+ = \coprod_I \tilde{\mathcal{Y}}_I$, and $\tilde{\mathcal{Y}}_m^+$ is open dense in $\tilde{\mathcal{X}}_m^+$, $\tilde{\mathcal{X}}_m^+ = \bigcup_I \overline{\tilde{\mathcal{Y}}_I}$ gives a decomposition into irreducible components, where $\overline{\tilde{\mathcal{Y}}_I}$ is the closure of $\tilde{\mathcal{Y}}_I$ in $\tilde{\mathcal{X}}_m^+$. Hence in order to prove the lemma, it is enough to show that $\tilde{\mathcal{X}}_I$ is closed in $\tilde{\mathcal{X}}_m^+$ for each I . But one can check that the closure Z_I of $\tilde{\mathcal{X}}_I$ in $\tilde{\mathcal{X}}$ is contained in the set $\tilde{\mathcal{X}}_I \cup \bigcup_{I'} \tilde{\mathcal{X}}_{I'}$, where I' runs over the subsets of $[1, n]$ such that $|I'| < m$. Hence $\tilde{\mathcal{X}}_I = Z_I \cap \tilde{\mathcal{X}}_m^+$ is closed in $\tilde{\mathcal{X}}_m^+$. \square

4.5. For a fixed m , we define a variety \mathcal{G}_m by

$$\mathcal{G}_m = \{(x, v, W) \mid (x, v) \in G^{u\theta} \times V, W : \text{isotropic subspace of } V, \\ \dim W = m, x(W) = W, v \in W\}.$$

We define a map $\pi_m : \tilde{\mathcal{X}}_m^+ \rightarrow G^{u\theta} \times V$ by the restriction of π . Then π_m can be decomposed as

$$\pi_m : \tilde{\mathcal{X}}_m^+ \xrightarrow{\varphi'} \mathcal{G}_m \xrightarrow{\varphi''} G^{u\theta} \times V$$

where $\varphi' : (x, v, gB^\theta) \mapsto (x, v, W(x, v))$, $\varphi'' : (x, v, W) \mapsto (x, v)$. Let us consider the spaces $W_0 = M_m$ and $\overline{V}_0 = W_0^\perp / W_0$. We put $G_1 = GL(W_0)$, $G_2 = GL(\overline{V}_0)$. Then \overline{V}_0 has a natural symplectic structure, and G_2 is identified with a θ -stable subgroup of G . Put $H_0 = G_1 \times G_2^\theta$. We define a variety

$$\mathcal{H}_m = \{(x, v, W, \phi_1, \phi_2) \mid (x, v, W) \in \mathcal{G}_m, \\ \phi_1 : W \xrightarrow{\sim} W_0, \phi_2 : W^\perp / W \xrightarrow{\sim} \overline{V}_0 \text{ (symplectic isom.)} \},$$

and morphisms

$$q : \mathcal{H}_m \rightarrow \mathcal{G}_m, \quad (x, v, W, \phi_1, \phi_2) \mapsto (x, v, W), \\ \sigma : \mathcal{H}_m \rightarrow G_1 \times G_2^{u\theta}, \quad (x, v, W, \phi_1, \phi_2) \mapsto (\phi_1(x|_W)\phi_1^{-1}, \phi_2(x|_{W^\perp/W})\phi_2^{-1}).$$

Then $H \times H_0$ acts freely on \mathcal{H}_m by

$$(g, (h_1, h_2)) : (x, v, W, \phi_1, \phi_2) \mapsto (gxg^{-1}, gv, g(W), h_1\phi_1g^{-1}, h_2\phi_2g^{-1})$$

for $g \in H$, $(h_1, h_2) \in H_0$. Moreover, σ is $H \times H_0$ -equivariant with respect to the natural action of H_0 and the trivial action of H on $G_1 \times G_2^{u\theta}$. We have

(4.5.1) The map q is a principal bundle with fibre isomorphic to H_0 .

(4.5.2) The map σ is a principal bundle with fibre isomorphic to H .

In fact, (4.5.1) is clear. We show (4.5.2). Since H acts freely on \mathcal{H}_m , preserving each fibre, it is enough to show that each fibre consists of a single H -orbit. For a fixed $(x', x'') \in G_1 \times G_2^{u\theta}$, the fibre $\sigma^{-1}(x', x'')$ is determined by the following process;

(i) choose an isotropic subspace W of V such that $\dim W = m$,

(ii) for each W , choose an isomorphism $\phi_1 : W \rightarrow W_0$ and a symplectic isomorphism $\phi_2 : W^\perp/W \rightarrow \overline{V}_0$,

(iii) choose $x \in G^{\iota\theta}$ such that $\phi_1(x|_W)\phi_1^{-1} = x'$, $\phi_2(x|_{W^\perp/W})\phi_2^{-1} = x''$,

(iv) choose $v \in W$.

Let P be the stabilizer of the flag $(W_0 \subset W_0^\perp)$ in G . Then P is θ -stable, and is decomposed as $P = LU_P$, where L is a θ -stable Levi subgroup of P containing T and U_P is the unipotent radical of P . For (i), such W are parametrized by H/P^θ . For (ii), they are parametrized by $G_1 \times G_2^\theta$. For (iii), x should be contained in $P^{\iota\theta}$, but x', x'' determines the part corresponding to $L^{\iota\theta}$. Hence the choice of x is parametrized by $U_P^{\iota\theta}$. Finally, for (iv), v is any element in W . It follows that each fibre $\sigma^{-1}(x', x'')$ is irreducible with the same dimension as H , and so it consists of a single H -orbit. Hence (4.5.2) follows.

Let B_1 is a Borel subgroup of G_1 which is the stabilizer of the flag $(M_k)_{0 \leq k \leq m}$ in G_1 , and B_2 a θ -stable Borel subgroup of G_2 which is the stabilizer of the flag $(M_{m+1}/M_m \subset \cdots \subset M_m^\perp/M_m)$ in G_2 . Put

$$\begin{aligned}\tilde{G}_1 &= \{(x, gB_1) \in G_1 \times G_1/B_1 \mid g^{-1}xg \in B_1\}, \\ \tilde{G}_2^{\iota\theta} &= \{(x, gB_2) \in G_2^{\iota\theta} \times G_2^\theta/B_2^\theta \mid g^{-1}xg \in B_2^{\iota\theta}\},\end{aligned}$$

and define maps $\pi^1 : \tilde{G}_1 \rightarrow G_1$, $\pi^2 : \tilde{G}_2^{\iota\theta} \rightarrow G_2^{\iota\theta}$ by first projections. Thus π^2 is the map π given in 1.8 with respect to $G_2^{\iota\theta}$, and π^1 is the corresponding map for G_1 . We define a variety

$$\begin{aligned}\mathcal{Z}_m^+ &= \{(x, v, gB^\theta, \phi_1, \phi_2) \mid (x, v, gB^\theta) \in \tilde{\mathcal{X}}_m^+, \\ &\quad \phi_1 : W(x, v) \xrightarrow{\sim} W_0, \phi_2 : W(x, v)^\perp/W(x, v) \xrightarrow{\sim} \overline{V}_0\},\end{aligned}$$

and a map $\tilde{q} : \mathcal{Z}_m^+ \rightarrow \mathcal{X}_m^+$ by a natural projection. We define a map $\tilde{\sigma} : \mathcal{Z}_m^+ \rightarrow \tilde{G}_1 \times \tilde{G}_2^{\iota\theta}$ as follows; take $(x, v, gB^\theta, \phi_1, \phi_2) \in \mathcal{Z}_m^+$. By the construction of $W(x, v)$, $W(x, v)$ is contained in $g(M_n)$, and $(g(M_k))_{0 \leq k \leq n}$ is an x -stable isotropic flag in V . Then $(W(x, v) \cap g(M_k))$ gives an x -stable flag $(W_k)_{0 \leq k \leq m}$ in $W(x, v)$, and $(W(x, v) + g(M_k)/W(x, v))$ gives an x -stable isotropic flag $(\overline{V}_\ell)_{0 \leq \ell \leq n-m}$ in $W(x, v)^\perp/W(x, v)$. We denote by $g_1 B_1 g_1^{-1}$ the stabilizer of the flag $(\phi_1(W_k))$ in G_1 , and by $g_2 B^\theta x_1 g_2^{-1}$ the stabilizer of the isotropic flag $(\phi_2(\overline{V}_\ell))$ in \overline{V}_0 . The map $\tilde{\sigma}$ is defined as

$$(x, v, gB^\theta, \phi_1, \phi_2) \mapsto (\phi_1(x|_{W(x, v)})\phi_1^{-1}, g_1 B_1), (\phi_2(x|_{W(x, v)^\perp/W(x, v)})\phi_2^{-1}, g_2 B_2^\theta).$$

We consider a diagram

$$(4.5.3) \quad \begin{array}{ccccc} T_1 \times T_2^{\iota\theta} & \xleftarrow{f} & T^{\iota\theta} & \xrightarrow{\text{id}} & T^{\iota\theta} \\ \alpha^1 \times \alpha^2 \uparrow & & \uparrow \tilde{\alpha} & & \uparrow \alpha \\ \tilde{G}_1 \times \tilde{G}_2^{\iota\theta} & \xleftarrow{\tilde{\sigma}} & \mathcal{Z}_m^+ & \xrightarrow{\tilde{q}} & \tilde{\mathcal{X}}_m^+ \\ \pi^1 \times \pi^2 \downarrow & & \downarrow \tilde{\varphi}' & & \downarrow \varphi' \\ G_1 \times G_2^{\iota\theta} & \xleftarrow{\sigma} & \mathcal{H}_m & \xrightarrow{q} & \mathcal{G}_m \\ & & & & \downarrow \varphi'' \\ & & & & G^{\iota\theta} \times V, \end{array}$$

where the maps $\tilde{\varphi}'$, $\tilde{\alpha}$ are defined naturally. Note that $T^{\iota\theta}$ can be written as $T^{\iota\theta} \simeq T_1 \times T_2^{\iota\theta}$, where T_1 is a maximal torus of G_1 (the diagonal group), and T_2 is a θ -stable maximal torus of G_2 (the diagonal group). We fix an isomorphism $f : T^{\iota\theta} \rightarrow T_1 \times T_2^{\iota\theta}$. The maps $\alpha^1 : \tilde{G}_1 \rightarrow T_1$, $\alpha^2 : \tilde{G}_2^{\iota\theta} \rightarrow G_2^{\iota\theta}$ are defined as in 1.15.

4.6. Let \mathcal{E} be a tame local system on $T^{\iota\theta}$. Under the isomorphism $f : T^{\iota\theta} \rightarrow T_1 \times T_2^{\iota\theta}$, \mathcal{E} can be written as $\mathcal{E} \simeq \mathcal{E}_1 \boxtimes \mathcal{E}_2$, where \mathcal{E}_1 (resp. \mathcal{E}_2) is a tame local system on T_1 (resp. $T_2^{\iota\theta}$). Then we have $\mathcal{W}_{\mathbf{m}, \mathcal{E}} \simeq (S_m)_{\mathcal{E}_1} \times (S_{n-m})_{\mathcal{E}_2}$, where $(S_m)_{\mathcal{E}_1}$ is the stabilizer of \mathcal{E}_1 in $S_m \simeq \mathcal{W}_1 = N_{G_1}(T_1)/T_1$, and $(S_{n-m})_{\mathcal{E}_2}$ is the stabilizer of \mathcal{E}_2 in $S_{n-m} \simeq \mathcal{W}_2 = N_{G_2^{\iota\theta}}(T_2^{\iota\theta})/Z_{G_2^{\iota\theta}}(T_2^{\iota\theta})$. Let $\mathcal{A}_{\mathcal{E}_2}$ be the algebra defined in 1.15, by replacing $G^{\iota\theta}$ by $G_2^{\iota\theta}$, and let $\mathcal{A}_{\mathcal{E}_1}$ be the corresponding algebra for G_1 . Then we have $\mathcal{A}_{\mathbf{m}, \mathcal{E}} \simeq \mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}_2}$.

For $\rho_1 \in \mathcal{A}_{\mathcal{E}_1}^\wedge$, $\rho_2 \in \mathcal{A}_{\mathcal{E}_2}^\wedge$, we consider the complexes

$$K_1 = \text{IC}(G_1, \mathcal{L}_{\rho_1})[\dim G_1], \quad K_2 = \text{IC}(G_2^{\iota\theta}, \mathcal{L}_{\rho_2})[\dim G_2^{\iota\theta}].$$

Then $K_1 \boxtimes K_2$ is an H_0 -equivariant simple perverse sheaf on $G_1 \times G_2^{\iota\theta}$, and so $\sigma^*(K_1 \boxtimes K_2)[2n^2 + n]$ is an H_0 -equivariant simple perverse sheaf on \mathcal{H}_m by (4.5.2). Since q is a principal bundle with group H_0 by (4.5.1), there exists a unique simple perverse sheaf A_ρ on \mathcal{G}_m such that

$$q^*(A_\rho)[m^2 + 2(n-m)^2 + (n-m)] \simeq \sigma^*(K_1 \boxtimes K_2)[2n^2 + n],$$

where $\rho = \rho_1 \otimes \rho_2 \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}^\wedge$ under the isomorphism $\mathcal{A}_{\mathbf{m}, \mathcal{E}} \simeq \mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}_2}$.

Note that the map $\varphi' : \tilde{\mathcal{X}}_m^+ \rightarrow \mathcal{G}_m$ is not surjective. The image $\varphi'(\tilde{\mathcal{X}}_m^+)$ is described as follows; for each $m' \leq m$, we define a subvariety $\mathcal{G}_{m, m'}$ of \mathcal{G}_m by

$$\mathcal{G}_{m, m'} = \{(x, v, W) \in \mathcal{G}_m \mid \dim W(x, v) = m'\}.$$

Then $\mathcal{G}_m = \coprod_{0 \leq m' \leq m} \mathcal{G}_{m, m'}$ gives a partition of \mathcal{G}_m , and the closure of $\mathcal{G}_{m, m'}$ is a union of $\mathcal{G}_{m, m''}$ with $m'' \leq m'$. Put $\mathcal{G}_m^0 = \mathcal{G}_{m, m}$. Then \mathcal{G}_m^0 is an open dense subset of \mathcal{G}_m . It is clear from the definition of φ' that $\varphi'(\tilde{\mathcal{X}}_m^+)$ coincides with \mathcal{G}_m^0 . We denote

by A_ρ^0 the restriction of A_ρ on \mathcal{G}_m^0 , which gives rise to a simple perverse sheaf if the support of A_ρ has a non-trivial intersection with \mathcal{G}_m^0 .

Let \mathcal{L}_ρ be the local system on \mathcal{Y}_m^0 as in (4.1.1). Since \mathcal{Y}_m^0 is an open dense subset of \mathcal{X}_m^0 , one can consider the intersection cohomology $\mathrm{IC}(\mathcal{X}_m^0, \mathcal{L}_\rho)$. We show the following lemma.

Lemma 4.7. *Under the notation as above, we have*

$$\varphi''_* A_\rho^0 \simeq \mathrm{IC}(\mathcal{X}_m^0, \mathcal{L}_\rho)[\dim \mathcal{X}_m].$$

Proof. Since $\mathcal{G}_m^0 = \{(x, v, W) \in \mathcal{G}_m \mid W = W(x, v)\}$, φ'' gives an isomorphism \mathcal{G}_m^0 onto \mathcal{X}_m^0 by (4.3.3). Hence $\varphi''_* A_\rho^0$ is a simple perverse sheaf if A_ρ^0 is so. Put $K = \varphi''_* A_\rho^0$ and $d = \dim \mathcal{X}_m^0 = \dim \mathcal{X}_m$. In order to prove the lemma, it is enough to show that

$$(4.7.1) \quad \mathcal{H}^{-d} K|_{\mathcal{Y}_m^0} \simeq \mathcal{L}_\rho.$$

We consider the following commutative diagram

$$(4.7.2) \quad \begin{array}{ccccc} T_1 \times T_2^{\iota\theta} & \xleftarrow{f} & T^{\iota\theta} & \xrightarrow{\mathrm{id}} & T^{\iota\theta} \\ \alpha_0^1 \times \alpha_0^2 \uparrow & & \uparrow \tilde{\alpha}_0 & & \uparrow \alpha_0 \\ \tilde{G}_{1,\mathrm{reg}} \times \tilde{G}_{2,\mathrm{reg}}^{\iota\theta} & \xleftarrow{\tilde{\sigma}_0} & \tilde{\mathcal{Z}}_m^0 & \xrightarrow{\tilde{q}_0} & \tilde{\mathcal{Y}}_m^0 \\ \xi^1 \times \xi^2 \downarrow & & \downarrow \tilde{\xi}_0 & & \downarrow \xi_0 \\ (G_1/T_1 \times T_{1,\mathrm{reg}}) \times (G_2^\theta/Z_{G_2^\theta}(T_2^{\iota\theta}) \times T_{2,\mathrm{reg}}^{\iota\theta}) & \xleftarrow{\hat{\sigma}_0} & \hat{\mathcal{Z}}_m^0 & \xrightarrow{\hat{q}_0} & \hat{\mathcal{Y}}_m^0 \\ \eta^1 \times \eta^2 \downarrow & & \downarrow \tilde{\eta}_0 & & \downarrow \eta_0 \\ G_{1,\mathrm{reg}} \times G_{2,\mathrm{reg}}^{\iota\theta} & \xleftarrow{\sigma_0} & \mathcal{H}_{m,\mathrm{reg}} & \xrightarrow{q_0} & \mathcal{G}_{m,\mathrm{reg}}, \\ & & & & \downarrow \varphi''_0 \\ & & & & \mathcal{Y}_m^0 \end{array}$$

where $\tilde{\mathcal{Y}}_m^0 = \tilde{\mathcal{Y}}_I$, $\hat{\mathcal{Y}}_m^0 = \hat{\mathcal{Y}}_I$ for $I = [1, m]$ (cf. 3.1, 3.4), and

$$\begin{aligned} \tilde{G}_{1,\mathrm{reg}} &= (\pi^1)^{-1}(G_{1,\mathrm{reg}}), \\ \tilde{G}_{2,\mathrm{reg}}^{\iota\theta} &= (\pi^2)^{-1}(G_{2,\mathrm{reg}}), \\ \mathcal{G}_{m,\mathrm{reg}} &= (\varphi'')^{-1}(\mathcal{Y}_m^0), \\ \mathcal{H}_{m,\mathrm{reg}} &= q^{-1}(\mathcal{G}_{m,\mathrm{reg}}), \\ \tilde{\mathcal{Z}}_m^0 &= \tilde{q}^{-1}(\tilde{\mathcal{Y}}_m^0), \\ \hat{\mathcal{Z}}_m^0 &= \{(g * (t, v), \phi_1, \phi_2) \mid g * (t, v) \in \hat{\mathcal{Y}}_m^0, \\ &\quad \phi_1 : g(W_0) \xrightarrow{\sim} W_0, \phi_2 : g(W_0)^\perp / g(W_0) \xrightarrow{\sim} \overline{V}_0\}, \end{aligned}$$

and the maps $\tilde{q}_0, q_0, \tilde{\sigma}_0, \sigma_0$ are defined as the restrictions of the corresponding maps $\tilde{q}, q, \tilde{\sigma}, \sigma$. The map $\xi_0 = \xi_I$ for $I = [1, m]$. Note that in this case, $W(t, v) = M_m$ for $(t, v) \in T_{\text{reg}}^{\iota\theta} \times M_m^0$. We define a map $\eta_0 : \hat{\mathcal{Y}}_m^0 \rightarrow \mathcal{G}_{m, \text{reg}}$ by $(g * (t, v)) \mapsto (gtg^{-1}, g(v), g(M_m))$, and the maps $\tilde{\xi}_0, \tilde{\eta}_0$ are defined accordingly. The maps in the first column are defined by applying the discussion in 1.8 to the case $\psi^2 : \tilde{G}_{2, \text{reg}}^{\iota\theta} \rightarrow G_{2, \text{reg}}^{\iota\theta}$, and to the GL case $\psi^1 : \tilde{G}_{1, \text{reg}} \rightarrow G_{1, \text{reg}}$. The map $\hat{\sigma}_0$ is defined as follows; for $(g * (t, v), \phi_1, \phi_2) \in \hat{\mathcal{Z}}_m^0$, gtg^{-1} stabilizes $g(W_0)$, and $t'_1 = \phi_1(gtg^{-1}|_{g(W_0)})\phi_1^{-1} \in G_{1, \text{reg}}$, $t'_2 = \phi_2(gtg^{-1}|_{g(W_0)^\perp/g(W_0)})\phi_2^{-1} \in G_{2, \text{reg}}^{\iota\theta}$. We have $Z_{G_1}(t'_1) = g_1 T_1 g_1^{-1}$ with $g_1 \in G_1$, and $Z_{G_2^\theta}(t'_2) = g_2 Z_{G^\theta}(T_2^{\iota\theta}) g_2^{-1}$ with $g_2 \in G_2^\theta$. Then $\hat{\sigma}_0(g * (t, v), \phi_1, \phi_2) = ((g_1 T_1, t_1), (g_2 Z_{G_2^\theta}(T_2^{\iota\theta}), t_2))$, where $(t_1, t_2) = f(t) \in T_1 \times T_2^{\iota\theta}$.

It follows from the commutative diagram that

$$(4.7.3) \quad \tilde{\sigma}_0^*((\alpha_0^1)^* \mathcal{E}_1 \boxtimes (\alpha_0^2)^* \mathcal{E}_2) \simeq \tilde{q}_0^* \alpha_0^* \mathcal{E}.$$

One can check that the squares in the middle row and the lower row are all cartesian squares. Since $\psi^1 = \eta^1 \circ \xi^1$, $\psi^2 = \eta^2 \circ \xi^2$, we see that

$$(4.7.4) \quad \sigma_0^*(\psi_*^1(\alpha_0^1)^* \mathcal{E}_1 \boxtimes \psi_*^2(\alpha_0^2)^* \mathcal{E}_2) \simeq q_0^*(\eta_0 \circ \xi_0)_* \alpha_0^* \mathcal{E}.$$

Now $\psi_*^2(\alpha_0^2)^* \mathcal{E}_2$ is decomposed as in (1.15.1), and similarly for $\psi_*^1(\alpha_0^1)^* \mathcal{E}_1$. On the other hand, since

$$\mathcal{G}_{m, \text{reg}} = \{(x, v, W) \mid (x, v) \in \mathcal{Y}_m^0, W = W(x, v)\} \simeq \mathcal{Y}_m^0,$$

φ_0'' is an isomorphism. It follows, by (3.4.3) and (3.5.1) that

$$(\eta_0 \circ \xi_0)_* \alpha_0^* \mathcal{E} \simeq \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}^\wedge} H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho \otimes \mathcal{L}_\rho$$

under the identification $\mathcal{G}_{m, \text{reg}} \simeq \mathcal{Y}_m^0$. Since the decomposition by the Galois covering is compatible with σ_0 and q_0 by (4.7.2), we have $\sigma_0^*(\mathcal{L}_{\rho_1} \boxtimes \mathcal{L}_{\rho_2}) \simeq q_0^* \mathcal{L}_\rho$. This implies that $A_\rho^0|_{\mathcal{G}_{m, \text{reg}}} \simeq A_\rho|_{\mathcal{G}_{m, \text{reg}}} \simeq \mathcal{L}_\rho[d]$, and so

$$K|_{\mathcal{Y}_m^0} \simeq (\varphi_0'')_*(A_\rho^0|_{\mathcal{G}_{m, \text{reg}}}) \simeq \mathcal{L}_\rho[d].$$

This proves (4.7.1), and the lemma follows. \square

As a corollary to Lemma 4.7, we have the following.

Proposition 4.8. *Under the notation in Lemma 4.7, $(\pi_m)_* \alpha^* \mathcal{E}|_{\tilde{\mathcal{X}}_m^+}$ is a semisimple complex, which is a direct summand of*

$$\bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}^\wedge} H^\bullet(\mathbf{P}_1^{n-m}) \otimes \tilde{V}_\rho \otimes \text{IC}(\mathcal{X}_m^0, \mathcal{L}_\rho),$$

where $\tilde{V}_\rho = \mathcal{A}_\mathcal{E} \otimes_{\mathcal{A}_{\mathbf{m}, \mathcal{E}}} \rho$ is considered as a vector space, ignoring the $\mathcal{A}_\mathcal{E}$ -action.

Proof. First we recall the isomorphism given in 1.17.2 in the language of flags. Assume that $s \in T^{\iota\theta}$. Let $V = V_1 \oplus \cdots \oplus V_t$ be the decomposition of V to the eigenspaces of s . We assume that $V_i = \langle e_i, f_i \mid i \in J_i \rangle$, where $J_i = [n_1 + \cdots + n_{i-1} + 1, n_1 + \cdots + n_i]$. We regard J_i an ordered set under the natural ordering. Let $x = su$ be the Jordan decomposition x . Let $\mathcal{F}^\theta(V)$ be the set of isotropic flags on V , and $\mathcal{F}_x^\theta(V)$ the subset of isotropic flags stable by $x \in G^{\iota\theta}$, and similarly define $\mathcal{F}_x^\theta(V_i)$ for $x \in GL(V_i)^{\iota\theta}$. Let $\Gamma = W_{H,s} \backslash W_H$ be as in 1.17. We choose a set of distinguished representatives $\{w_\gamma \mid \gamma \in \Gamma\}$ for Γ , where w_γ is a permutation on the set $[1, n]$ such that w_γ^{-1} is order preserving on each subset J_i . Then $\mathcal{F}_s^\theta(V)$ is decomposed as $\mathcal{F}_s^\theta(V) = \coprod_{\gamma \in \Gamma} \mathcal{F}_s^\gamma(V)$, where $\mathcal{F}_s^\gamma(V)$ is the set of $Z_H(s)$ conjugates of the isotropic flag $\langle \langle e_{w_\gamma(1)} \rangle \subset \langle e_{w_\gamma(1)}, e_{w_\gamma(2)} \rangle \subset \cdots \subset \langle e_{w_\gamma(1)}, \dots, e_{w_\gamma(n)} \rangle \rangle$. For each flag $F^\bullet = (F_k) \in \mathcal{F}_s^\gamma(V)$, $F_1 \cap V_i \subseteq F_2 \cap V_i \subseteq \cdots$ gives an isotropic flag $F_i^\bullet \in \mathcal{F}^\theta(V_i)$, and the map $F^\bullet \mapsto (F_1^\bullet, \dots, F_t^\bullet)$ gives an isomorphism between $\mathcal{F}_s^\gamma(V)$ and $\prod_i \mathcal{F}^\theta(V_i)$. If we replace s by $x = su$ (Jordan decomposition) with $s \in T^{\iota\theta}$, and let u_i be the unipotent element in $GL(V_i)^{\iota\theta}$ obtained from u , then the above isomorphism restricts to the isomorphism between $\mathcal{F}_x^\theta(V)$ and $\prod_i \mathcal{F}_{u_i}^\theta(V_i)$.

We fix a subset $I \subset [1, n]$. For $x = su \in G^{\iota\theta}$ such that $s \in T^{\iota\theta}$, we consider the decomposition of V as above. Let $U = W(x, v)$. Then U is s -stable, and U is decomposed as $U = \bigoplus_i U_i$ with $U_i = U \cap V_i$. Thus there exists a subset J'_i of J_i such that $U_i = \langle e_i \mid i \in J'_i \rangle$. We now define a subset $\tilde{\mathcal{X}}_I^\spadesuit$ of $\tilde{\mathcal{X}}_m^+$ as the set of all H -conjugates of $(x, v, w_\gamma B^\theta) \in \tilde{\mathcal{X}}_m^+$ such that $|w_\gamma^{-1}(J_i) \cap I| = |J'_i|$ for each i , for all the possible choice of (x, v) with $s \in T^{\iota\theta}$ and $w_\gamma \in W_{s,H} \backslash W_H$. Then $\tilde{\mathcal{X}}_I^\spadesuit$ is a closed subset of $\tilde{\mathcal{X}}_m^+$ containing $\tilde{\mathcal{X}}_I$. (In fact, $\tilde{\mathcal{X}}_I$ coincides with the H -conjugates of $(x, v, w_\gamma B^\theta) \in \tilde{\mathcal{X}}_m^+$ such that $w_\gamma^{-1}(J_i) \cap I$ consists of the first $|J'_i|$ letters.) We have $\mathcal{G}_m^0 = \varphi'(\tilde{\mathcal{X}}_I^\spadesuit)$ for any I . Let $\varphi'_I : \tilde{\mathcal{X}}_I^\spadesuit \rightarrow \mathcal{G}_m^0$ be the restriction of φ' on $\tilde{\mathcal{X}}_I^\spadesuit$. We define a subset \mathcal{Z}_I^\spadesuit of \mathcal{Z}_m^+ by $\mathcal{Z}_I^\spadesuit = \tilde{q}^{-1}(\tilde{\mathcal{X}}_I^\spadesuit)$, and a subset \mathcal{H}_m^0 of \mathcal{H}_m by $\mathcal{H}_m^0 = q^{-1}(\mathcal{G}_m^0)$. We have a commutative diagram

$$(4.8.1) \quad \begin{array}{ccccc} \tilde{G}_1 \times \tilde{G}_2^{\iota\theta} & \xleftarrow{\tilde{\sigma}_I} & \mathcal{Z}_I^\spadesuit & \xrightarrow{\tilde{q}_I} & \tilde{\mathcal{X}}_I^\spadesuit \\ \pi^1 \times \pi^2 \downarrow & & \downarrow \tilde{\varphi}'_I & & \downarrow \varphi'_I \\ G_1 \times G_2^{\iota\theta} & \xleftarrow{\sigma_1} & \mathcal{H}_m^0 & \xrightarrow{q_1} & \mathcal{G}_m^0, \end{array}$$

where $q_1, \sigma_1, \tilde{q}_I, \tilde{\sigma}_I$ are restrictions of $q, \sigma, \tilde{q}, \tilde{\sigma}$, respectively. The square in the right hand side is cartesian since the corresponding square in (4.5.3) is cartesian. We note that

(4.8.2) The square in the left hand side is also cartesian.

Assume given $(x, v, U, \phi_1, \phi_2) \in \mathcal{H}_m^0$ and $(x_1, g_1 B_1) \in \tilde{G}_1$, $(x_2, g_2 B_2^\theta) \in \tilde{G}_2^\theta$, which is compatible with $\pi^1 \times \pi^2$ and σ_1 . We show that there exists a unique element $(x, v, g B^\theta, \phi_1, \phi_2) \in \mathcal{Z}_I^\spadesuit$ compatible with the above data with respect to $\tilde{\sigma}_I, \tilde{\varphi}'_I$. Let $x = su$ be the Jordan decomposition of x . Since $\sigma_1, \tilde{\sigma}_I$ are H -equivariant, we may assume that $s \in T^{\iota\theta}$, and that V_i 's are given as above. Let $U = W(x, v)$ and $U_i =$

$U \cap V_i$ be as before. Under the isomorphism $\phi_1 : U \xrightarrow{\sim} W_0, \phi_2 : U^\perp/U \xrightarrow{\sim} \overline{V}_0$, x_1, x_2 give rise to elements $x|_U, x_{U^\perp/U}$, which we denote also by x_1, x_2 . Under this isomorphism, $(x_1, g_1 B_1) \in \tilde{G}_1$ determines an x_1 -stable flag on U , which is given as $(D_1^\bullet, \dots, D_t^\bullet)$ with $D_i^\bullet \in \mathcal{F}_{u_i}(U_i)$ for each i . Similarly, $(x_2, g_2 B_2^\theta) \in \tilde{G}_2^\theta$ determines an x_2 -stable isotropic flag on U^\perp/U , which is given as $(E_1^\bullet, \dots, E_t^\bullet)$ with $E_i^\bullet \in \mathcal{F}_{u_i}^\theta(U_i^\perp/U_i)$, where U_i^\perp is defined with respect to V_i . Now the pair $(D_i^\bullet, E_i^\bullet)$ determines a unique flag $F_i^\bullet \in \mathcal{F}_{u_i}^\theta(V_i)$. Let s_1 (resp. s_2) be the semisimple part of x_1 (resp. x_2), and Γ_1 (resp. Γ_2) a similar set as Γ defined for $s_1 \in GL(U)$ (resp. $s_2 \in GL(U^\perp/U)^\theta$). Then $(x_1, g_1 B_1)$ determines $\gamma_1 \in \Gamma_1$, and $(x_2, g_2 B_2^\theta)$ determines $\gamma_2 \in \Gamma_2$. Here w_{γ_1} (resp. w_{γ_2}) is a permutation of the set J' (resp. J''), where $J' = \coprod_i J'_i$ and J'' is the complement of J' in $[1, n]$. Now the pair (γ_1, γ_2) determines a unique $\gamma \in \Gamma$ as follows; let τ be the permutation on $[1, n]$ which maps J' to I , and preserves the order on J' and on J'' . We define a permutation w by

$$w(j) = \begin{cases} \tau w_{\gamma_1}^{-1}(j) & \text{if } j \in J', \\ \tau w_{\gamma_2}^{-1}(j) & \text{if } j \in J''. \end{cases}$$

We take $\gamma = W_{H,s} w^{-1} \in W_{H,s} \backslash W_H$. Under the isomorphism $\prod_i \mathcal{F}_{u_i}^\theta(V_i) \simeq \mathcal{F}_x^\gamma(V)$ given above, $(F_1^\bullet, \dots, F_t^\bullet)$ determines a unique flag $F^\bullet \in \mathcal{F}_x^\gamma(V)$. Let gB^θ be the element in H/B^θ corresponding to F^\bullet . Then $(x, v, gB^\theta, \phi_1, \phi_2) \in \mathcal{Z}_I^\bullet$, and it satisfies our assertion. This proves (4.8.2).

We define a complex K_{T_2, \mathcal{E}_2} on G_2^θ as in (1.15.2), by replacing G by G_2 . A similar construction works for the group case G_1 . They are given as follows;

$$\begin{aligned} K_{T_1, \mathcal{E}_1} &= \bigoplus_{\rho_1 \in \mathcal{A}_{\mathcal{E}_1}^\wedge} \rho_1 \otimes \text{IC}(G_1, \mathcal{L}_{\rho_1})[\dim G_1], \\ K_{T_2, \mathcal{E}_2} &= \bigoplus_{\rho_2 \in \mathcal{A}_{\mathcal{E}_2}^\wedge} H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho_2 \otimes \text{IC}(G_2^\theta, \mathcal{L}_{\rho_2})[\dim G_2^\theta]. \end{aligned}$$

Then we have the following result, the first one is due to Lusztig [L2], and the second one follows from Theorem 1.16 due to Grojonowski.

$$\begin{aligned} \pi_*^1(\alpha^1)^* \mathcal{E}_1[\dim G_1] &\simeq K_{T_1, \mathcal{E}_1}, \\ \pi_*^2(\alpha^2)^* \mathcal{E}_2[\dim G_2^\theta] &\simeq K_{T_2, \mathcal{E}_2}. \end{aligned}$$

By using the diagram (4.8.1) together with 4.6, we have

$$(\varphi'_I)_* \alpha^* \mathcal{E}[\dim \mathcal{X}_m] \simeq \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}^\wedge} H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho \otimes A_\rho^0.$$

Since $\tilde{\mathcal{X}}_I$ is a connected component of $\tilde{\mathcal{X}}_I^\bullet$, $(\varphi'|_{\tilde{\mathcal{X}}_I})_*\alpha^*\mathcal{E}[\dim \mathcal{X}_m]$ is a direct summand of $(\varphi'_I)_*\alpha^*\mathcal{E}[\dim \mathcal{X}_m]$. Since $\tilde{\mathcal{X}}_m^+ = \coprod_I \tilde{\mathcal{X}}_I$, $(\varphi')_*\alpha^*\mathcal{E}[\dim \mathcal{X}_m]$ is a direct summand of

$$(4.8.3) \quad \bigoplus_{\substack{I \subset [1, n] \\ |I| = m}} \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \varepsilon}^\wedge} H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho \otimes A_\rho^0.$$

By applying φ''_* on (4.8.3), and by applying Lemma 4.7, we obtain the proposition. \square

4.9. We are now ready to prove Theorem 4.2. For each $0 \leq m \leq n$, let $\bar{\pi}_m$ be the restriction of π on $\pi^{-1}(\mathcal{X}_m)$. We show that the direct image complex $(\bar{\pi}_m)_*\alpha^*\mathcal{E}|_{\pi^{-1}(\mathcal{X}_m)}$ on \mathcal{X}_m is decomposed as follows;

$$(4.9.1) \quad \begin{aligned} (\bar{\pi}_m)_*(\alpha^*\mathcal{E}|_{\pi^{-1}(\mathcal{X}_m)})[d_m] &\simeq \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}, \varepsilon}^\wedge} H^\bullet(\mathbf{P}_1^{n-m}) \otimes \tilde{V}_\rho \otimes \mathrm{IC}(\mathcal{X}_m, \mathcal{L}_\rho)[d_m] \\ &\quad \oplus \bigoplus_{0 \leq m' < m} \bigoplus_{\rho \in \mathcal{A}_{\mathbf{m}', \varepsilon}} \tilde{V}_\rho \otimes \mathrm{IC}(\mathcal{X}_{m'}, \mathcal{L}_\rho)[d_{m'} - 2(n - m)]. \end{aligned}$$

Note that the theorem will follow from (4.9.1) by applying it to the case where $m = n$. We show (4.9.1) by induction on m . In the discussion below, we use the same symbol $\alpha^*\mathcal{E}$ to denote its restriction to $\pi^{-1}(\mathcal{X}_m)$, $\pi^{-1}(\mathcal{Y}_m)$, etc. for saving the notation. Since $\bar{\pi}_m$ is a proper map, by the decomposition theorem, $(\bar{\pi}_m)_*\alpha^*\mathcal{E}$ can be written as a direct sum of complexes of the form $A[h]$, where A is a simple perverse sheaf. Here the restriction of $(\bar{\pi}_m)_*\alpha^*\mathcal{E}$ on \mathcal{Y}_m coincides with $(\bar{\psi}_m)_*\alpha^*\mathcal{E}$, and we know by (3.6.1) that it can be expressed by a similar formula as (4.9.1), by replacing $\mathcal{X}_m, \mathcal{X}_{m'}$ by $\mathcal{Y}_m, \mathcal{Y}_{m'}$. Hence in order to prove (4.9.1), it is enough to show that $\mathrm{supp} A \cap \mathcal{Y}_m \neq \emptyset$ for any direct summand $A[h]$ of $(\bar{\pi}_m)_*\alpha^*\mathcal{E}$. We have $\mathcal{X}_m = \mathcal{X}_m^0 \cup \mathcal{X}_{m-1}$ with \mathcal{X}_m^0 open. The restriction of $(\bar{\pi}_m)_*\alpha^*\mathcal{E}$ on \mathcal{X}_m^0 (resp. \mathcal{X}_{m-1}) coincides with $(\pi_m)_*\alpha^*\mathcal{E}$ (resp. $(\pi_{m-1})_*\alpha^*\mathcal{E}$). Let $A[h]$ be a direct summand of $(\bar{\pi}_m)_*\alpha^*\mathcal{E}$. If $\mathrm{supp} A \cap \mathcal{X}_m^0 \neq \emptyset$, then $A|_{\mathcal{X}_m^0}$ is a direct summand (up to shift) of $(\pi_m)_*\alpha^*\mathcal{E}$. By comparing Proposition 4.8 and (3.5.2), we see that $\mathrm{supp} A \cap \mathcal{Y}_m \neq \emptyset$. If $\mathrm{supp} A \cap \mathcal{X}_m^0 = \emptyset$, then $\mathrm{supp} A \subset \mathcal{X}_{m-1}$, and $A|_{\mathcal{X}_{m-1}}$ (up to shift) is a direct summand of $(\pi_{m-1})_*\alpha^*\mathcal{E}$. By induction hypothesis, we see again that $\mathrm{supp} A \cap \mathcal{Y}_m \neq \emptyset$. This proves (4.9.1) and so proves the theorem.

5. SPRINGER CORRESPONDENCE

5.1. The Springer correspondence is a bijective correspondence between the set of H -orbits in $G_{\mathrm{uni}}^\theta \times V$ and the set of irreducible representations (up to isomorphism) of Weyl group W_n , which is an analogy of the Springer correspondence in the case of reductive groups. Kato proved in [Ka1] the Springer correspondence for the exotic nilpotent cone $\mathfrak{g}_{\mathrm{nil}}^{-\theta} \times V$ over \mathbf{C} . Here we give an alternate proof for $\mathcal{X}_{\mathrm{uni}} = G_{\mathrm{uni}}^\theta \times V$ over an algebraically closed field \mathbf{k} of odd characteristic, based on Theorem 4.2.

In the setup of Section 4, we consider the special case where $\mathcal{E} = \bar{\mathbf{Q}}_l$, the constant sheaf on $T^{\iota\theta}$. In this case, $\mathcal{A}_{\mathcal{E}}$ is isomorphic to the group algebra $\bar{\mathbf{Q}}_l[\mathcal{W}]$, and $\bar{\mathcal{A}}_{\mathcal{E}}$ is isomorphic to $\bar{\mathbf{Q}}_l[W_n]$. Hence by Theorem 4.2, $\pi_*\bar{\mathbf{Q}}_l$ is equipped with the action of W_n . It follows that for each $(x, v) \in \mathcal{X}$, $H^i(\pi^{-1}(x, v), \bar{\mathbf{Q}}_l)$ turns out to be a W_n -module. In the case where $(x, v) = (1, 0)$, $\pi^{-1}(x, v) = H/B^{\theta}$. Hence W_n acts on $H^i(H/B^{\theta}, \bar{\mathbf{Q}}_l)$, which we call the exotic action of W_n . On the other hand, W_n acts on $H^i(H/B^{\theta}, \bar{\mathbf{Q}}_l)$ as the original Springer action of Weyl groups, which we call the classical action. We have the following lemma.

Lemma 5.2. *The exotic action and the classical action of W_n on $H^i(H/B^{\theta}, \bar{\mathbf{Q}}_l)$ coincide with each other.*

Proof. We prove the lemma following the argument analogous to the classical case (see e.g. [S1, Proposition 5.4]). Let $\pi' : \tilde{G}^{\iota\theta} \rightarrow G^{\iota\theta}$, $\psi' : \pi'^{-1}(G_{\text{reg}}^{\iota\theta}) \rightarrow G_{\text{reg}}^{\iota\theta}$ be the maps defined in 1.8. Under the embedding $G^{\iota\theta} \simeq G^{\iota\theta} \times \{0\} \hookrightarrow G^{\iota\theta} \times V$, $\pi_*\bar{\mathbf{Q}}_l|_{G^{\iota\theta}} \simeq \pi'_*\bar{\mathbf{Q}}_l$ and $\psi_*\bar{\mathbf{Q}}_l|_{G_{\text{reg}}^{\iota\theta}} \simeq \psi'_*\bar{\mathbf{Q}}_l$. Moreover, $\psi_*\bar{\mathbf{Q}}_l|_{G_{\text{reg}}^{\iota\theta}}$ (resp. $\pi_*\bar{\mathbf{Q}}_l|_{G^{\iota\theta}}$) coincides with $(\psi_m)_*\bar{\mathbf{Q}}_l$ (resp. $(\pi_m)_*\bar{\mathbf{Q}}_l$) defined in 3.4 (resp. 4.5) for $m = 0$. Thus the arguments in Section 3 and 4 are applied for this situation. In particular, $\pi'_*\bar{\mathbf{Q}}_l$ and $\psi'_*\bar{\mathbf{Q}}_l$ have W_n -actions as a restriction of $\psi_*\bar{\mathbf{Q}}_l$ and $\pi_*\bar{\mathbf{Q}}_l$. Thus, forgetting the vector space part V , we have only to consider $\pi'_*\bar{\mathbf{Q}}_l$. We can also consider the Lie algebra counter part of $\pi'_*\bar{\mathbf{Q}}_l$. Let $\pi'' : \tilde{\mathfrak{g}}^{-\theta} \rightarrow \mathfrak{g}^{-\theta}$ be the map analogous to π' , where $\tilde{\mathfrak{g}}^{-\theta} = H \times^{B^{\theta}} \mathfrak{b}^{-\theta}$ with $\mathfrak{b} = \text{Lie } B$. Then one can define an action of W_n on $\pi''_*\bar{\mathbf{Q}}_l$ by a similar construction. Consider the map $\log : G^{\iota\theta} \rightarrow \mathfrak{g}^{-\theta}$. Then \log maps $B^{\iota\theta}$ to $\mathfrak{b}^{-\theta}$, and it induces a map $\widehat{\log} : \tilde{G}^{\iota\theta} \rightarrow \tilde{\mathfrak{g}}^{-\theta}$, which is a base change of \log via π'' . Thus we have $\log^*\pi'_*\bar{\mathbf{Q}}_l \simeq \pi''_*\bar{\mathbf{Q}}_l$. This isomorphism is W_n -equivariant. Hence in order to prove the lemma, we may consider the case $\pi'' : \tilde{\mathfrak{g}}^{-\theta} \rightarrow \mathfrak{g}^{-\theta}$. For saving the notation, we use the same symbol in the Lie algebra case as in 1.8, such as $\pi : \tilde{\mathfrak{g}}^{-\theta} \rightarrow \mathfrak{g}^{-\theta}$.

Let $\mathfrak{t} = \text{Lie } T$, and $\mathfrak{g}_{\text{reg}}^{-\theta}, \mathfrak{t}_{\text{reg}}^{-\theta}$ be as in Proposition 1.14. Put $\mathfrak{b}_{\text{reg}}^{-\theta} = \mathfrak{b} \cap \mathfrak{g}_{\text{reg}}^{-\theta}$. Similar to (1.8.1), we have a diagram

$$\psi : H \times^{B^{\theta}} \mathfrak{b}_{\text{reg}}^{-\theta} \simeq H \times^{B^{\theta} \cap Z_H(T^{\iota\theta})} \mathfrak{t}_{\text{reg}}^{-\theta} \xrightarrow{\xi} H \times^{Z_H(T^{\iota\theta})} \mathfrak{t}_{\text{reg}}^{-\theta} \xrightarrow{\eta} \mathfrak{g}_{\text{reg}}^{-\theta},$$

where ξ is a locally trivial fibration with fibre isomorphic to \mathbf{P}_1^n , and η is a finite Galois covering with group S_n . Let \mathcal{O} be an H -orbit in $\mathfrak{g}_{\text{reg}}^{-\theta}$, and put $\tilde{\mathcal{O}} = \xi^{-1}\eta^{-1}(\mathcal{O})$. We have the following diagram

$$(5.2.1) \quad \begin{array}{ccccccc} \psi^{-1}(\mathcal{O}) & \xleftarrow{\sim} & \tilde{\mathcal{O}} & \xrightarrow{\xi} & \eta^{-1}(\mathcal{O}) & \xrightarrow{\eta} & \mathcal{O} \\ q \downarrow & & \downarrow \tilde{p} & & \downarrow p & & \\ H/B^{\theta} & \xleftarrow{\gamma} & H/B^{\theta} \cap Z_H(T^{\iota\theta}) & \xrightarrow{\xi'} & H/Z_H(T^{\iota\theta}), & & \end{array}$$

where q, \tilde{p}, p are natural projections. Now $\psi_*\bar{\mathbf{Q}}_l$ is equipped with W_n -action. Hence for any subvariety X of $\mathfrak{g}_{\text{reg}}^{-\theta}$, W_n acts on $H^i(\psi^{-1}(X), \bar{\mathbf{Q}}_l) \simeq \mathbf{H}^i(X, \psi_*\bar{\mathbf{Q}}_l)$, which is the exotic action of W_n . Moreover, for the inclusion $X \hookrightarrow Y$, the induced map

$H^i(\psi^{-1}(Y), \bar{\mathbf{Q}}_l) \rightarrow H^i(\psi^{-1}(X), \bar{\mathbf{Q}}_l)$ is W_n -equivariant. Here we fix an action of W_n on $H^i(\psi^{-1}(\mathcal{O}), \bar{\mathbf{Q}}_l)$ defined as above. First we note that

(5.2.2) The map $q^* : H^i(H/B^\theta, \bar{\mathbf{Q}}_l) \rightarrow H^i(\psi^{-1}(\mathcal{O}), \bar{\mathbf{Q}}_l)$ is W_n -equivariant with respect to the exotic action of W_n on $H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$.

In fact, let j be the inclusion map $\psi^{-1}(\mathcal{O}) \hookrightarrow \tilde{\mathfrak{g}}^{-\theta}$, and $q_1 : \tilde{\mathfrak{g}}^{-\theta} \rightarrow H/B^\theta$ be the projection. Then $q = q_1 \circ j$. Since j^* is W_n -equivariant, it is enough to show that q_1^* is W_n -equivariant. But since $\tilde{\mathfrak{g}}^{-\theta}$ is a vector bundle over H/B^θ , $H^i(\tilde{\mathfrak{g}}^{-\theta}, \bar{\mathbf{Q}}_l) \simeq H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$. This implies that $q_1^* = (j_1^*)^{-1}$, where j_1 is a natural inclusion $H/B^\theta \hookrightarrow \tilde{\mathfrak{g}}^{-\theta}$ so that $q_1 \circ j_1 = \text{id}$. Since j_1^* is W_n -equivariant, q_1^* is W_n -equivariant.

Let $\mathcal{W} = N_H(T^{\iota\theta})/Z_H(T^{\iota\theta})$ be as before. Then \mathcal{W} acts on $H/Z_H(T^{\iota\theta})$, and p is \mathcal{W} -equivariant. It follows that $p^* : H^i(H/Z_H(T^{\iota\theta}), \bar{\mathbf{Q}}_l) \rightarrow H^i(\eta^{-1}(\mathcal{O}), \bar{\mathbf{Q}}_l)$ is \mathcal{W} -equivariant. $B^\theta \cap Z_H(T^{\iota\theta}) \simeq B_2 \times \cdots \times B_2$ as in 1.8, and the action of S_n (as a permutation group of the set $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$) on H leaves $B^\theta \cap Z_H(T^{\iota\theta})$ invariant. Hence $\mathcal{W} \simeq S_n$ acts on $H/B^\theta \cap Z_H(T^{\iota\theta})$, and ξ' is \mathcal{W} -equivariant. Since ξ, ξ' are both \mathbf{P}_1^n -bundles, we see that the map $\tilde{p}^* : H^i(H/B^\theta \cap Z_H(T^{\iota\theta}), \bar{\mathbf{Q}}_l) \rightarrow H^i(\tilde{\mathcal{O}}, \bar{\mathbf{Q}}_l)$ is \mathcal{W} -equivariant. The group \mathcal{W} acts on $B^\theta \cap Z_H(T^{\iota\theta})$ as a permutation of factors B_2 , and its restriction on T^θ coincides with the permutation action of S_n on the diagonal groups. Since γ is a vector bundle, we have $H^i(H/B^\theta, \bar{\mathbf{Q}}_l) \simeq H^i(H/T^\theta, \bar{\mathbf{Q}}_l) \simeq H^i(H/B^\theta \cap Z_H(T^{\iota\theta}), \bar{\mathbf{Q}}_l)$, and γ^* becomes \mathcal{W} -equivariant, where $\mathcal{W} \simeq S_n$ acts naturally on $H^i(H/T^\theta, \bar{\mathbf{Q}}_l)$, which is the classical action on $H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$. This shows that

(5.2.3) q^* is S_n -equivariant with respect to the classical action of S_n on $H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$.

On the other hand, $\xi_* \bar{\mathbf{Q}}_l \simeq H^\bullet(\mathbf{P}_1^n) \otimes \bar{\mathbf{Q}}_l$ is equipped with the action of $(\mathbf{Z}/2\mathbf{Z})^n$. Thus $\mathbf{H}^i(\eta^{-1}(\mathcal{O}), \xi_* \bar{\mathbf{Q}}_l) \simeq H^i(\tilde{\mathcal{O}}, \bar{\mathbf{Q}}_l)$ has a structure of $(\mathbf{Z}/2\mathbf{Z})^n$ -module. Similarly, $H^i(H/B^\theta \cap Z_H(T^{\iota\theta}), \bar{\mathbf{Q}}_l)$ has a structure of $(\mathbf{Z}/2\mathbf{Z})^n$ -module, and the map \tilde{p}^* is $(\mathbf{Z}/2\mathbf{Z})^n$ -equivariant. Let $\xi_1 : H/T^\theta \rightarrow H/Z_H(T^{\iota\theta})$. Then $H/B^\theta \cap Z_H(T^{\iota\theta})$ is a vector bundle over H/T^θ , and

$$H^i(H/B^\theta \cap Z_H(T^{\iota\theta}), \bar{\mathbf{Q}}_l) \simeq H^i(H/T^\theta, \bar{\mathbf{Q}}_l) \simeq \mathbf{H}^i(H/Z_H(T^{\iota\theta}), (\xi_1)_* \bar{\mathbf{Q}}_l).$$

Here the $(\mathbf{Z}/2\mathbf{Z})^n$ -action on $H^i(H/B^\theta \cap Z_H(T^{\iota\theta}), \bar{\mathbf{Q}}_l)$ comes from the $(\mathbf{Z}/2\mathbf{Z})^n$ -action on $(\xi_1)_* \bar{\mathbf{Q}}_l$, which is described as follows: $Z_H(T^{\iota\theta}) \simeq SL_2 \times \cdots \times SL_2$, and $Z_H(T^{\iota\theta})/T^\theta \simeq (SL_2/T_2)^n$, where T_2 is a maximal torus of SL_2 . Hence $(\mathbf{Z}/2\mathbf{Z})^n$ acts on $Z_H(T^{\iota\theta})/T^\theta$ as its Weyl group, and acts on $H/T^\theta \simeq H^{Z_H(T^{\iota\theta})}(Z_H(T^{\iota\theta})/T^\theta)$ naturally, which gives the $(\mathbf{Z}/2\mathbf{Z})^n$ -action on $(\xi_1)_* \bar{\mathbf{Q}}_l$. But this action of $(\mathbf{Z}/2\mathbf{Z})^n$ on H/T^θ is nothing but the action on H/T^θ as a subgroup of $W_n = N_H(T^\theta)/T^\theta$. It follows that

(5.2.4) q^* is $(\mathbf{Z}/2\mathbf{Z})^n$ -equivariant with respect to the classical action of $(\mathbf{Z}/2\mathbf{Z})^n$ on $H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$.

By (5.2.3) and (5.2.4), we see that q^* is W_n -equivariant with respect to the classical action of W_n on $H^i(H/B^\theta, \bar{\mathbf{Q}}_l)$. In view of (5.2.2), in order to prove the lemma, it is enough to show that q^* is injective. But since $\mathcal{O} \simeq H/Z_H(T^{\iota\theta})$, and

η is a finite covering with group S_n , we see that $\eta^{-1}(\mathcal{O}) = \coprod_{w \in S_n} X_w$ with $X_w \simeq H/Z_H(T^{\iota\theta})$. Since ξ is a \mathbf{P}_1^n -bundle, $\tilde{\mathcal{O}}$ is written as $\tilde{\mathcal{O}} = \coprod_{w \in S_n} \tilde{X}_w$, where \tilde{X}_w is a \mathbf{P}_1^n -bundle over X_w , which is isomorphic to $H/B^\theta \cap Z_H(T^{\iota\theta})$. This implies the injectivity of the map \tilde{p}^* . The injectivity of q^* follows from this. Thus the lemma is proved. \square

5.3. Note that $\dim \mathcal{X}_{\text{uni}} = 2\nu_H$ by Proposition 2.7. Then π_1 is semismall by Lemma 2.5 (ii), and so $K_1 = (\pi_1)_* \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\text{uni}}]$ is a semisimple perverse sheaf. As π_1 is H -equivariant, K_1 is H -equivariant. Since the number of H -orbits in \mathcal{X}_{uni} is finite, an H -equivariant simple perverse sheaf is of the form $A_\mu = \text{IC}(\bar{\mathcal{O}}_\mu, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_\mu]$ for an H -orbit \mathcal{O}_μ in \mathcal{X}_{uni} . It follows that

$$(5.3.1) \quad K_1 \simeq \bigoplus_{\mu \in \mathcal{P}_{n,2}} \rho_\mu \otimes A_\mu,$$

where $\rho_\mu = \text{Hom}(A_\mu, K_1)$ is a W_n -module.

Since $\mathcal{E} = \bar{\mathbf{Q}}_l$, we have $\mathcal{A}_{\mathbf{m}, \mathcal{E}} \simeq \bar{\mathbf{Q}}_l[S_m \times S_{n-m}]$. Then $\rho \in \mathcal{A}_{\mathbf{m}, \mathcal{E}}^\wedge$ is given by $\rho_1 \otimes \rho_2$, where $\rho_1 \in S_m^\wedge$ (resp. $\rho_2 \in S_{n-m}^\wedge$) corresponding to $\mu^{(1)} \in \mathcal{P}_m$ (resp. $\mu^{(2)} \in \mathcal{P}_{n-m}$). The induced representation \tilde{V}_ρ gives rise to an irreducible representation of W_n , which we denote by \tilde{V}_μ for $\mu = (\mu^{(1)}, \mu^{(2)}) \in \mathcal{P}_{n,2}$. In this case we denote the local system \mathcal{L}_ρ on \mathcal{Y}_m^0 by \mathcal{L}_μ . For a given $\mu \in \mathcal{P}_{n,2}$, put $m(\mu) = |\mu^{(1)}|$. Hence \mathcal{L}_μ is a local system on $\mathcal{Y}_{m(\mu)}^0$.

The following result is a reformulation of Kato's result ([Ka1, Theorem 8.3]).

Theorem 5.4 (Springer correspondence). (i) *For each $\mu \in \mathcal{P}_{n,2}$, ρ_μ is an irreducible W_n -module. The map $\mathcal{O}_\mu \mapsto \rho_\mu$ gives a bijective correspondence between the set of H -orbits in \mathcal{X}_{uni} and the set of irreducible representations (up to isomorphism) of W_n .*

(ii) *Let $\mathcal{O}_{\mu^\bullet}$ be the H -orbit, for a certain $\mu^\bullet \in \mathcal{P}_{n,2}$, corresponding to \tilde{V}_μ under the above correspondence. Then we have*

$$\text{IC}(\mathcal{X}_{m(\mu)}, \mathcal{L}_\mu)|_{\mathcal{X}_{\text{uni}}} \simeq \text{IC}(\bar{\mathcal{O}}_{\mu^\bullet}, \bar{\mathbf{Q}}_l)[a],$$

where $a = \dim \mathcal{O}_{\mu^\bullet} - \dim \mathcal{X}_{\text{uni}} - \dim \mathcal{X}_{m(\mu)} + \dim \mathcal{X}$.

Proof. We show (i). We have an algebra homomorphism

$$(5.4.1) \quad \bar{\mathbf{Q}}_l[W_n] \xrightarrow{\alpha} \text{End } K_1 \xrightarrow{\beta} \text{End } H^*(H/B^\theta, \bar{\mathbf{Q}}_l).$$

In order to prove (i), it is enough to show that α is an isomorphism $\bar{\mathbf{Q}}_l[W_n] \xrightarrow{\sim} \text{End } K_1$. Via $\beta \circ \alpha$, W_n acts on $H^*(H/B^\theta, \bar{\mathbf{Q}}_l)$. By Lemma 5.2, this action is the classical action of W_n . Since W_n acts on $H^*(H/B^\theta, \bar{\mathbf{Q}}_l)$ as the regular representation, we see that $\beta \circ \alpha$ is injective. Hence α is also injective, and we have $|W_n| \leq \dim \text{End } K_1$.

On the other hand, (5.3.1) implies that

$$(5.4.2) \quad H^i(\pi_1^{-1}(x, v), \bar{\mathbf{Q}}_l) \simeq \bigoplus_{\mu \in \mathcal{P}_{n,2}} \rho_\mu \otimes \mathcal{H}_{(x,v)}^{i-\dim \mathcal{X}_{\text{uni}}+\dim \mathcal{O}_\mu} \text{IC}(\bar{\mathcal{O}}_\mu, \bar{\mathbf{Q}}_l).$$

Take $(x, v) \in \mathcal{O}_\mu$ and assume that $\rho_\mu \neq \{0\}$. Put $d_\mu = (\dim \mathcal{X}_{\text{uni}} - \dim \mathcal{O}_\mu)/2 = \nu_H - \dim \mathcal{O}_\mu/2$. Then (5.4.2) implies that

$$(5.4.3) \quad H^{2d_\mu}(\pi_1^{-1}(x, v), \bar{\mathbf{Q}}_l) \simeq \rho_\mu \otimes \mathcal{H}_{(x,v)}^0 \text{IC}(\bar{\mathcal{O}}_\mu, \bar{\mathbf{Q}}_l) \simeq \rho_\mu.$$

By applying Lemma 2.5 (ii), we have $\dim \pi_1^{-1}(x, v) = d_\mu$. Hence $\dim \rho_\mu$ coincides with the number of irreducible components of $\pi_1^{-1}(x, v)$ with dimension d_μ . Thus by Lemma 2.5 (iii), we have

$$\dim \text{End } K_1 = \sum_{\mu \in \mathcal{P}_{n,2}} (\dim \rho_\mu)^2 \leq \sum_{\mu \in \mathcal{P}_{n,2}} c_\mu^2 \leq |W_n|.$$

It follows that α is an isomorphism, and (i) follows.

Next we show (ii). It follows from Theorem 4.2, we have

$$\pi_* \bar{\mathbf{Q}}_l[\dim \mathcal{X}]|_{\mathcal{X}_{\text{uni}}} \simeq \bigoplus_{\mu \in \mathcal{P}_{n,2}} \tilde{V}_\mu \otimes \text{IC}(\mathcal{X}_{m(\mu)}, \mathcal{L}_\mu)[d_{m(\mu)}]|_{\mathcal{X}_{\text{uni}}}.$$

On the other hand, by (i) and (5.3.1), we have

$$(5.4.4) \quad (\pi_1)_* \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\text{uni}}] \simeq \bigoplus_{\mu \in \mathcal{P}_{n,2}} \tilde{V}_\mu \otimes \text{IC}(\bar{\mathcal{O}}_{\mu^\bullet}, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_{\mu^\bullet}].$$

By comparing these two formulas, we obtain (ii). Thus the theorem is proved. \square

5.5. Let $P = LU_P$ be a θ -stable parabolic subgroup of G containing B , where L is a θ -stable Levi subgroup containing T . Then L^θ is isomorphic to $GL_{n_1} \times \cdots \times GL_{n_k} \times Sp_{2n'}$ with $\sum_i n_i + n' = n$. Accordingly, we have a decomposition $M_n = \bigoplus_{i \geq 0} V_i$ and $M'_n = \bigoplus_{i \geq 0} V'_i$, where V_i, V'_i are T -stable subspaces with $\dim V_i = \dim V'_i = n_i$ for $i \geq 1$, and $\dim V_0 = \dim V'_0 = n'$. Let $V_L = V_0 \oplus V'_0$ and $\tilde{V}_L = V_L \oplus V_1 \oplus \cdots \oplus V_k$ so that $V = \tilde{V}_L \oplus V'_L$, where $V'_L = V'_1 \oplus \cdots \oplus V'_k$. Here we may assume that V'_L is U_P -stable, and \tilde{V}_L is identified with V/V'_L , on which L^θ acts naturally. We consider the variety

$$(5.5.1) \quad L^\theta \times \tilde{V}_L \simeq (GL_{2n'}^\theta \times V_L) \times \prod_{i=1}^k (GL_{n_i} \times V_i),$$

on which L^θ acts via the conjugation action on L^θ and the natural action on \tilde{V}_L . In particular, $GL_{2n'}^\theta \times V_L$ is a similar variety as $H \times V$ for $n = n'$. It is known by [AH], [T] that the set of GL_{n_i} -orbits on $(GL_{n_i})_{\text{uni}} \times V_{n_i}$ is parametrised by the set

$\mathcal{P}_{n_i,2}$. Let $\pi_P : P_{\text{uni}}^{\iota\theta} \times V \rightarrow L_{\text{uni}}^{\iota\theta} \times \tilde{V}_L$ be the map defined by the natural projection $P_{\text{uni}}^{\iota\theta} \rightarrow L_{\text{uni}}^{\iota\theta}$ on the first factor, and the projection $\tilde{V}_L \oplus V'_L \rightarrow \tilde{V}_L$ on the second factor. Let \mathcal{O} be an H -orbit in $G_{\text{uni}}^{\iota\theta} \times V$, and \mathcal{O}_L an L^θ -orbit in $L_{\text{uni}}^{\iota\theta} \times V_L$. Let us consider a variety

$$\begin{aligned} \mathcal{Z} = \{ & (x, v, gP^\theta, g'P^\theta) \in G_{\text{uni}}^{\iota\theta} \times V \times H/P^\theta \times H/P^\theta \\ & | (g^{-1}xg, g^{-1}v) \in \pi_P^{-1}(\mathcal{O}_L), (g'^{-1}xg', g'^{-1}v) \in \pi_P^{-1}(\mathcal{O}_L) \}. \end{aligned}$$

We consider the partition $H/P^\theta \times H/P^\theta = \coprod X_\omega$ into H -orbits, and let $\mathcal{Z}_\omega = p^{-1}(X_\omega)$, where $p : \mathcal{Z} \rightarrow H/P^\theta \times H/P^\theta$ is the projection onto the last two factors. Recall that $\nu_H = \dim U^\theta$. Here we put $\bar{\nu} = \nu_{L^\theta}$, the number of positive roots for L^θ . Let $c = \dim \mathcal{O}$ and $\bar{c} = \dim \mathcal{O}_L$. Note that c is even by Lemma 2.3, however, \bar{c} is not necessarily even since a GL_{n_i} -orbit on $(GL_{n_i})_{\text{uni}} \times V_i$ is not so in general (see [AH, Proposition 2.8]). The following result, which is a generalization of Proposition 2.5, is an analogy of the dimension formula due to Lusztig [L2, Proposition 1.2], and can be proved in a similar line.

Proposition 5.6. (i) For $(\bar{x}, \bar{v}) \in \mathcal{O}_L$, we have $\dim(\mathcal{O} \cap \pi_P^{-1}(\bar{x}, \bar{v})) \leq (c - \bar{c})/2$.
(ii) For $(x, v) \in \mathcal{O}$,

$$\dim\{gP^\theta \in H/P^\theta \mid (g^{-1}xg, g^{-1}v) \in \pi_P^{-1}(\mathcal{O}_L)\} \leq (\nu_H - c/2) - (\bar{\nu} - \bar{c}/2).$$

(iii) Put $d_0 = 2\nu_H - 2\bar{\nu} + \bar{c}$. Then $\dim \mathcal{Z}_\omega \leq d_0$ for all ω . Hence $\dim \mathcal{Z} \leq d_0$.

Proof. The proposition makes sense for groups of the same type as L , and we prove it for those groups. Since the assertion holds in the case where $G = T$, a maximal torus, we may assume that the proposition holds for a proper subgroup L of G . Consider the map $p : \mathcal{Z}_\omega \rightarrow X_\omega$. X_ω is an H -orbit of (P^θ, wP^θ) , where $w \in W_n$ is a representative of the double coset $W_P \backslash W_n / W_P$ for a parabolic subgroup W_P of W_n . Hence in order to show (iii), it is enough to see that

$$\begin{aligned} (5.6.1) \quad & \dim\{(x, v) \in \pi_P^{-1}(\mathcal{O}_L) \mid (\dot{w}^{-1}x\dot{w}, \dot{w}^{-1}v) \in \pi_P^{-1}(\mathcal{O}_L)\} \\ & \leq 2\nu_H - 2\bar{\nu} + \bar{c} - \dim X_\omega, \end{aligned}$$

where $\dot{w} \in H$ is a representative of w . Then an element $x \in P^{\iota\theta} \cap {}^wP^{\iota\theta}$ can be written as $x = \ell u = \ell' u'$ with $\ell \in L^{\iota\theta}$, $\ell' \in {}^wL^{\iota\theta}$, $u \in U_P^{\iota\theta}$, $u' \in {}^wU_P^{\iota\theta}$. Moreover, there exists $z \in L^{\iota\theta} \cap {}^wL^{\iota\theta}$ such that $\ell = zy'$, $\ell' = zy$ with $y' \in L^{\iota\theta} \cap {}^wU_P^{\iota\theta}$, $y \in {}^wL^{\iota\theta} \cap U_P^{\iota\theta}$. Hence (5.6.1) can be rewritten as

$$\begin{aligned} (5.6.2) \quad & \dim\{(v, u, u', y, y', z) \in V \times U_P^{\iota\theta} \times {}^wU_P^{\iota\theta} \\ & \times ({}^wL \cap U_P)^{\iota\theta} \times (L \cap {}^wU_P)^{\iota\theta} \times (L \cap {}^wL)^{\iota\theta} \\ & | y'u = yu', (zy', \bar{v}) \in \mathcal{O}_L, (\dot{w}^{-1}zy\dot{w}, \overline{\dot{w}^{-1}v}) \in \mathcal{O}_L\} \leq 2\nu_H - 2\bar{\nu} + \bar{c} - \dim X_\omega, \end{aligned}$$

where $\bar{v} \in \tilde{V}_L$ is the image of v under the map $V \rightarrow \tilde{V}_L$, and similarly for $\overline{\dot{w}^{-1}v} \in \tilde{V}_L$. For a fixed (y, y') , the variety $\{(u, u') \in (U_P \times {}^wU_P)^{\iota\theta} \mid y'u = yu'\}$ has dimension $\dim(U_P \cap {}^wU_P)^{\iota\theta}$. On the other hand, for a fixed $z, y, y', \bar{v}, \overline{\dot{w}^{-1}v}$,

$$\dim\{v \in V \mid (zy', \bar{v}) \in \mathcal{O}_L, (\dot{w}^{-1}zy\dot{w}, \overline{\dot{w}^{-1}v}) \in \mathcal{O}_L\} = \dim(V'_L \cap w(V'_L)).$$

Here we choose $w \in W_P \backslash W_n / W_P$ as a distinguished representative. Then we have $\dim(V'_L \cap w(V'_L)) = \#\{e_i \in V'_L \mid w^{-1}(e_i) \in V'_L\}$, which we set b_w . On the other hand, if we denote the set of positive roots of type C_n by $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid (1 \leq i < j \leq n), 2\varepsilon_i \mid (1 \leq i \leq n)\}$ and $\Delta_{P^\theta}^+$ the subset of Δ^+ corresponding to U_{P^θ} , we see that $\dim(U_P \cap {}^wU_P)^{\iota\theta} = \dim(U_P \cap {}^wU_P)^{\iota\theta} + c_w$, where

$$c_w = \#\{2\varepsilon_i \in \Delta_{P^\theta}^+ \mid w^{-1}(2\varepsilon_i) \in \Delta_{P^\theta}^+\}.$$

It is easy to see that $c_w = b_w$. Since $L \cap {}^wP$ is a parabolic subgroup of L with a Levi decomposition $L \cap {}^wP = (L \cap {}^wL)(L \cap {}^wU_P)$, and similarly for ${}^wL \cap P = ({}^wL \cap L)({}^wL \cap U_P)$, we see that $\dim(P \cap {}^wP)^\theta = 2\bar{\nu} + \dim T^\theta + \dim(U_P \cap {}^wU_P)^\theta$. It follows that

$$\dim(U_P \cap {}^wU_P)^{\iota\theta} + b_w = 2\nu_H - 2\bar{\nu} - \dim X_\omega.$$

Hence (5.6.2) will follow if we can show that

$$(5.6.3) \quad \dim\{(z, y, y', \bar{v}) \in (L \cap {}^wL)^{\iota\theta} \times ({}^wL \cap U_P)^{\iota\theta} \times (L \cap {}^wU_P)^{\iota\theta} \times \tilde{V}_L \mid (zy', \bar{v}) \in \mathcal{O}_L, (\dot{w}^{-1}zy\dot{w}, \dot{w}^{-1}\bar{v}) \in \mathcal{O}_L\} \leq \bar{c}.$$

Since wP and L contain a common maximal torus T , $Q = L \cap {}^wP$ is a θ -stable parabolic subgroup of L with Levi decomposition $Q = MU_Q$, where $M = L \cap {}^wL$, $U_Q = L \cap {}^wU_P$. Hence by replacing G, P, L by L, Q, M , one can define a map $\pi_Q : Q_{\text{uni}}^{\iota\theta} \times \tilde{V}_L \rightarrow M_{\text{uni}}^{\iota\theta} \times \tilde{V}_M$ as in 5.5. Similarly, for a parabolic subgroup $Q' = {}^wL \cap P$ of wL , the map $\pi_{Q'} : (Q')_{\text{uni}}^{\iota\theta} \times \tilde{V}_L \rightarrow M_{\text{uni}}^{\iota\theta} \times \tilde{V}_M$ is defined. Now $M_{\text{uni}}^{\iota\theta} \times \tilde{V}_M$ is partitioned into finitely many M^θ -orbits $\hat{\mathcal{O}}_1, \dots, \hat{\mathcal{O}}_m$, and for $(x', v') \in \hat{\mathcal{O}}_i$, the variety

$$\{(z, y, y', \bar{v}) \mid (zy', \bar{v}) \in \mathcal{O}_L \cap \pi_Q^{-1}(x', v'), (zy, \bar{v}) \in \mathcal{O}_w \cap \pi_{Q'}^{-1}(x', v')\}$$

is isomorphic to the product of two varieties appeared in Proposition 5.6 (i) by replacing G, L by $L, L \cap {}^wL$, and by ${}^wL, L \cap {}^wL$. Hence by induction hypothesis, its dimension is smaller than $(\bar{c} - \dim \hat{\mathcal{O}}_i)/2 + (\bar{c} - \dim \hat{\mathcal{O}}_i)/2$. It follows that

$$(5.6.4) \quad \dim\{(z, y, y', \bar{v}) \mid (z, \bar{v}') \in \hat{\mathcal{O}}_i, (zy', \bar{v}) \in \mathcal{O}_L, (zy, \bar{v}) \in \mathcal{O}_w\} \leq \bar{c},$$

where \bar{v}' is the image of \bar{v} under the map $\tilde{V}_L \rightarrow \tilde{V}_M$. Since (5.6.4) holds for any i , we obtain (5.6.3), and so (iii) holds.

We show (ii). Let q be the projection $\mathcal{Z} \rightarrow G_{\text{uni}}^{\iota\theta} \times V$ to the first two factors. For each \mathcal{O} , we consider $q^{-1}(\mathcal{O})$. If $q^{-1}(\mathcal{O})$ is empty, then the variety in (ii) is also empty, and the assertion holds. So we assume that $q^{-1}(\mathcal{O})$ is not empty. By (iii), we

have $\dim q^{-1}(\mathcal{O}) \leq d_0$. For each $(x, v) \in \mathcal{O}$, $q^{-1}(x, v)$ is a product of two varieties isomorphic to the variety given in (ii). Hence the dimension of the variety in (ii) is equal to $(\dim q^{-1}(\mathcal{O}) - \dim \mathcal{O})/2 \leq (d_0 - c)/2 = \nu_H - \bar{\nu} - c/2 + \bar{c}/2$. Thus (ii) follows.

We show (i). Consider the variety

$$\mathcal{X}_{\mathcal{O}} = \{((x, v), gP^{\theta}) \in \mathcal{O} \times H/P^{\theta} \mid (g^{-1}xg, g^{-1}v) \in \pi_P^{-1}(\mathcal{O}_L)\},$$

and let f be the projection on the \mathcal{O} -factors. Then for each $(x, v) \in \mathcal{O}$, the fibre $f^{-1}(x, v)$ is isomorphic to the variety given in (ii), hence the dimension of $\mathcal{X}_{\mathcal{O}}$ is less than or equal to $\nu_H - \bar{\nu} + c/2 + \bar{c}/2$. If we project it to the H/P^{θ} -factor, its fibre is isomorphic to the variety $\mathcal{O} \cap \pi_P^{-1}(\mathcal{O}_L)$. Hence

$$\dim \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}_L) \leq \nu_H - \bar{\nu} + c/2 + \bar{c}/2 - \dim H/P^{\theta} = (c + \bar{c})/2.$$

Now π_P maps the variety $\mathcal{O} \cap \pi_P^{-1}(\mathcal{O}_L)$ onto \mathcal{O}_L , and each fibre is isomorphic to the variety given in (i). Hence its dimension $\leq (c + \bar{c})/2 - \bar{c} = (c - \bar{c})/2$. This proves (i), and the proposition follows. \square

6. RESTRICTION THEOREM

6.1. In this section we shall prove the restriction theorem for Springer representations, which plays a crucial role for determining the Springer correspondence explicitly. Let $P = LU_P$ be as in 5.5. Here we consider the special case where $L^{\theta} \simeq GL_1 \times Sp_{2n-2}$. Let $V = \tilde{V}_L \oplus V'_L$ be the decomposition in 5.5. Take $z = (x, v) \in G_{\text{uni}}^{\theta} \times V$ and $z' = (x', v') \in L_{\text{uni}}^{\theta} \times \tilde{V}_L$. Here we assume that $v' \in V_L$. Then U_P stabilizes $v' + V'_L$. We define a variety $Y_{z, z'}$ by

$$Y_{z, z'} = \{g \in H \mid g^{-1}xg \in x'U_P^{\theta}, g^{-1}v \in v' + V'_L\}.$$

The group $Z_H(z)$ and $Z_{L^{\theta}}(z')U_P^{\theta}$ acts on $Y_{z, z'}$ by $(g_0, g_1)g = g_0gg_1^{-1}$.

Let $d_{z, z'} = (\dim Z_H(z) - \dim Z_{L^{\theta}}(z'))/2 + \dim U_P^{\theta}$. By Proposition 5.6 (ii), we have $\dim Y_{z, z'} \leq d_{z, z'}$. Let $S_{z, z'}$ be the set of irreducible components of $Y_{z, z'}$ of dimension $d_{z, z'}$.

Let $W = W_n$ be the Weyl group of H and W_L be the Weyl subgroup of W corresponding to L^{θ} . Then $W_L \simeq W_{n-1}$, where W_{n-1} is embedded in W_n as the subgroup with respect to $n - 1$ letters $\{2, \dots, n\}$. For $z \in G_{\text{uni}}^{\theta} \times V$, we denote by ρ_z^G the irreducible representation of W corresponding to the H -orbit containing z under the Springer correspondence (Theorem 5.4). Note that the variety $L_{\text{uni}}^{\theta} \times V_L$ with L^{θ} -action is exactly the same as $G_{\text{uni}}^{\theta} \times V$ with Sp_{2n} -action, by replacing n by $n - 1$. Hence one can consider the Springer correspondence with respect to W_L . For $z' \in L_{\text{uni}}^{\theta} \times V_L$, we denote by $\rho_{z'}^L$ the irreducible representation of W_L corresponding to z' . We shall prove the following theorem, which is an analogy of [L2, Theorem 8.3].

Theorem 6.2 (restriction theorem). *For $z \in G_{\text{uni}}^{\iota\theta} \times V$, $z' \in L_{\text{uni}}^{\iota\theta} \times V_L$, we have*

$$\langle \text{Res}_{W_L}^W \rho_z^G, \rho_{z'}^L \rangle_{W_L} = |S_{z,z'}|.$$

6.3. The theorem will be proved in 6.7 after some preliminaries. Let \tilde{L} be the subgroup of G generated by L and $B \cap Z_G(T^{\iota\theta})$. Then \tilde{L} is θ -stable, and $\tilde{L}^\theta \simeq (B_2 \times L^\theta)/T_2$, where under the isomorphism $Z_G(T^{\iota\theta}) \simeq GL_2 \times \cdots \times GL_2$, B_2 is the Borel subgroup of SL_2 corresponding to the first factor (contained in B), and T_2 is a maximal torus of B_2 contained in L^θ . We define varieties

$$\begin{aligned} \tilde{\mathcal{Y}}^L &= \{(x, v, g\tilde{L}^\theta) \in G_{\text{reg}}^{\iota\theta} \times V \times H/\tilde{L}^\theta \mid (g^{-1}xg, g^{-1}v) \in \mathcal{Y}^L\}, \\ \mathcal{Y}^L &= \bigcup_{g \in \tilde{L}^\theta} g(T_{\text{reg}}^{\iota\theta} \times M_n). \end{aligned}$$

Then the map $\psi : \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ is decomposed as

$$\psi : \tilde{\mathcal{Y}} \xrightarrow{\psi'} \tilde{\mathcal{Y}}^L \xrightarrow{\psi''} \mathcal{Y},$$

where $\psi' : (x, v, g(B^\theta \cap Z_H(T^{\iota\theta}))) \mapsto (x, v, g\tilde{L}^\theta)$ under the second expression in (3.1.2), and $\psi'' : (x, v, g\tilde{L}^\theta) \mapsto (x, v)$. Following the notations in 3.1, 3.2, for each $I \subset [1, n]$ such that $|I| = m$, we also define varieties $\tilde{\mathcal{Y}}_I^L, \mathcal{Y}_I^L$;

$$\begin{aligned} \tilde{\mathcal{Y}}_I^L &= \{(x, v, g\tilde{L}^\theta) \in G_{\text{reg}}^{\iota\theta} \times V \times H/\tilde{L}^\theta \mid (g^{-1}xg, g^{-1}v) \in \mathcal{Y}_I^L\} \simeq H \times^{\tilde{L}^\theta} \mathcal{Y}_I^L, \\ \mathcal{Y}_I^L &= \bigcup_{g \in \tilde{L}^\theta} g(T_{\text{reg}}^{\iota\theta} \times M_I). \end{aligned}$$

Note that in the case where $m \geq 1$, \mathcal{Y}_I^L coincides with $\mathcal{Y}_{[1,m]}^L$ if $1 \in I$, and coincides with $\mathcal{Y}_{[2,m+1]}^L$ if $1 \notin I$. Then $\tilde{\mathcal{Y}}_I^L$ coincides with $\psi'(\tilde{\mathcal{Y}}_I)$ for any I , and coincides with $\tilde{\mathcal{Y}}_{[1,m]}^L$ or $\tilde{\mathcal{Y}}_{[2,m+1]}^L$ according to the case where $1 \in I$ or $1 \notin I$ if $|I| \neq 0$. Now the map $\psi_I : \tilde{\mathcal{Y}}_I \rightarrow \mathcal{Y}_m^0$ is decomposed as

$$(6.3.1) \quad \psi_I : \tilde{\mathcal{Y}}_I \xrightarrow{\psi'_I} \tilde{\mathcal{Y}}_I^L \xrightarrow{\psi''_I} \mathcal{Y}_m^0,$$

where ψ'_I is the restriction of ψ' on $\tilde{\mathcal{Y}}_I$, and ψ''_I is the restriction of ψ'' on $\tilde{\mathcal{Y}}_I^L$. Put $\hat{\mathcal{Y}}_I^L = H \times^{Z_H(T^{\iota\theta})_J} (T_{\text{reg}}^{\iota\theta} \times M_I)$, where $J = I \cup \{1\}$. Then $Z_H(T^{\iota\theta})_J \subset \tilde{L}^\theta$ and one can define a map $\eta'_I : \hat{\mathcal{Y}}_I^L \rightarrow \tilde{\mathcal{Y}}_I^L$ by $(x, v, gZ_H(T^{\iota\theta})_J) \mapsto (x, v, g\tilde{L}^\theta)$. Now ψ'_I is decomposed as

$$(6.3.2) \quad \psi'_I : \tilde{\mathcal{Y}}_I \xrightarrow{\xi'_I} \hat{\mathcal{Y}}_I^L \xrightarrow{\eta'_I} \tilde{\mathcal{Y}}_I^L,$$

where $\xi'_I : (x, v, g(B^\theta \cap Z_H(T^{\iota\theta})) \mapsto (x, v, gZ_H(T^{\iota\theta})_J)$. Here η'_I is a finite Galois covering with group $\mathcal{W}_{I,L}$, where $\mathcal{W}_{I,L}$ is a subgroup of $\mathcal{W}_L \simeq S_{n-1}$ such that $\mathcal{W}_{I,L} \simeq S_{I-\{1\}} \times S_{I'}$ if $J = I$, and $\mathcal{W}_{I,L} \simeq S_I \times S_{I'-\{1\}}$ if $J \neq I$. Moreover, ξ'_I is a \mathbf{P}_1^{n-m} -bundle (resp. \mathbf{P}_1^{n-m-1} -bundle) if $J = I$ (resp. $J \neq I$). It follows that $\text{End}((\eta'_I)_* \bar{\mathbf{Q}}_I) \simeq \bar{\mathbf{Q}}_I[\mathcal{W}_{I,L}]$ and we have

$$(6.3.3) \quad (\eta'_I)_* \bar{\mathbf{Q}}_I \simeq \bigoplus_{\rho' \in \mathcal{W}_{I,L}^\wedge} \rho' \otimes \mathcal{L}_{\rho'},$$

where $\mathcal{L}_{\rho'} = \text{Hom}_{\mathcal{W}_{I,L}}(\rho', (\eta'_I)_* \bar{\mathbf{Q}}_I)$ is an irreducible local system on $\tilde{\mathcal{Y}}_I^L$. This implies that

$$(6.3.4) \quad (\psi'_I)_* \bar{\mathbf{Q}}_I \simeq \bigoplus_{\rho' \in \mathcal{W}_{I,L}^\wedge} H^\bullet(\mathbf{P}_1^{n-m-\varepsilon}) \otimes \rho' \otimes \mathcal{L}_{\rho'},$$

where $\varepsilon = 0$ or 1 according to the case $J = I$ or $J \neq I$. (In the case where $I = \emptyset$, we understand that $\varepsilon = 1$ and $\mathcal{W}_{I,L} = \mathcal{W}_L$). Since $\psi_I = \psi''_I \circ \psi'_I$, we see that

$$(\psi_I)_* \bar{\mathbf{Q}}_I \simeq (\psi''_I)_* (\psi'_I)_* \bar{\mathbf{Q}}_I \simeq \bigoplus_{\rho' \in \mathcal{W}_{I,L}^\wedge} H^\bullet(\mathbf{P}_1^{n-m-\varepsilon}) \otimes \rho' \otimes (\psi''_I)_* \mathcal{L}_{\rho'}.$$

Since $\psi_I = \eta_I \circ \xi_I$, by using the decomposition (3.4.3) and (3.5.1), we have

$$(6.3.5) \quad (\psi''_I)_* \mathcal{L}_{\rho'} \simeq \bigoplus_{\rho \in \mathcal{W}_I^\wedge} H^\bullet(\mathbf{P}_1^\varepsilon) \otimes \text{Hom}_{\mathcal{W}_{I,L}}(\rho', \rho) \otimes \mathcal{L}_\rho.$$

We put $I_1 = [1, m]$ and $I_2 = [2, m+1]$. Let ψ'_m be the restriction of ψ' on $\tilde{\mathcal{Y}}_m^+$, and let $\psi''_m : \tilde{\mathcal{Y}}_m^{L,+} = (\psi'')^{-1}(\mathcal{Y}_m^0) \rightarrow \mathcal{Y}_m^0$ be the restriction of ψ'' . Then $\tilde{\mathcal{Y}}_m^{L,+} = \tilde{\mathcal{Y}}_{I_1}^L \amalg \tilde{\mathcal{Y}}_{I_2}^L$ gives the decomposition to irreducible components. We have

$$\psi_m : \tilde{\mathcal{Y}}_m^+ \xrightarrow{\psi'_m} \tilde{\mathcal{Y}}_m^{L,+} \xrightarrow{\psi''_m} \mathcal{Y}_m^0.$$

Note that $\mathcal{W}_L \simeq S_{n-1}$ is the stabilizer of 1 in \mathcal{W} under the identification $\mathcal{W} \simeq S_n$. Let $\tilde{\mathcal{W}}_L, \tilde{\mathcal{W}}_{I,L}$ be the corresponding subgroups of $\tilde{\mathcal{W}}$. Now (6.3.4) implies, by a similar argument as in 3.5, that

$$(6.3.6) \quad \begin{aligned} (\psi'_m)_* \bar{\mathbf{Q}}_I &\simeq \bigoplus_{\rho' \in \mathcal{W}_{I_1,L}^\wedge} \text{Ind}_{\tilde{\mathcal{W}}_{I_1,L}}^{\tilde{\mathcal{W}}_L} (H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho') \otimes \mathcal{L}_{\rho'} \\ &\oplus \bigoplus_{\rho' \in \mathcal{W}_{I_2,L}^\wedge} \text{Ind}_{\tilde{\mathcal{W}}_{I_2,L}}^{\tilde{\mathcal{W}}_L} (H^\bullet(\mathbf{P}_1^{n-m-1}) \otimes \rho') \otimes \mathcal{L}_{\rho'} \end{aligned}$$

for $m \geq 1$, where $\mathcal{L}_{\rho'}$ is a local system on $\tilde{\mathcal{Y}}_{I_\varepsilon}^L$ corresponding to $\rho' \in \mathcal{W}_{I_\varepsilon,L}^\wedge$ for $\varepsilon = 1$ or 2 . While $(\psi'_m)_* \bar{\mathbf{Q}}_I$ for $m = 0$ is given by $(\psi'_I)_* \bar{\mathbf{Q}}_I$ for $I = \emptyset$.

Now $\mathcal{W}_{I_1, L}^\wedge$ is parametrized by a subset of $\mathcal{P}_{n-1, 2}$ consisting of μ' such that $m(\mu') = m - 1$, and we denote $\mathcal{L}_{\rho'}$ by $\mathcal{L}_{\mu'}$ if $\rho' \in \mathcal{W}_{I_1, L}^\wedge$ corresponds to $\mu' \in \mathcal{P}_{n-1, 2}$. Let $\tilde{\mathcal{Y}}_m^L$ be the closure of $\tilde{\mathcal{Y}}_{[1, m]}^L$ in $\tilde{\mathcal{Y}}^L$. Then $\tilde{\mathcal{Y}}_m^L = \coprod_{m' < m} \tilde{\mathcal{Y}}_{m'}^{L, +} \coprod \tilde{\mathcal{Y}}_{[1, m]}^L$. (The closure of $\tilde{\mathcal{Y}}_{[2, m+1]}^L$ in $\tilde{\mathcal{Y}}^L$ is given by $\coprod_{1 \leq m' \leq m} \tilde{\mathcal{Y}}_{[2, m'+1]}^L$.) Put $d_m^L = \dim \tilde{\mathcal{Y}}_m^L$. Then $\tilde{\mathcal{Y}}_{[1, m]}^L$ is an open dense smooth subset of $\tilde{\mathcal{Y}}_m^L$, and we consider the intersection cohomology $\mathrm{IC}(\tilde{\mathcal{Y}}_m^L, \mathcal{L}_{\mu'})$. By using a similar argument as in the proof of Proposition 3.6, one can show that

$$(6.3.7) \quad \psi'_* \bar{\mathbf{Q}}_l[d_n^L] \simeq \bigoplus_{\mu' \in \mathcal{P}_{n-1, 2}} \tilde{V}_{\mu'} \otimes \mathrm{IC}(\tilde{\mathcal{Y}}_{m(\mu') + 1}^L, \mathcal{L}_{\mu'})[d_{m(\mu') + 1}^L],$$

where $d_n^L = \dim \tilde{\mathcal{Y}}^L$. Moreover, $\mathrm{IC}(\tilde{\mathcal{Y}}_{m(\mu') + 1}^L, \mathcal{L}_{\mu'})$ is a constructible sheaf on $\tilde{\mathcal{Y}}_{m(\mu') + 1}^L$.

On the other hand, in view of (6.3.5), we have

$$(\psi''_*) \mathcal{L}_{\rho'} \simeq \begin{cases} \bigoplus_{\rho \in \mathcal{W}_{I_1}^\wedge} \mathrm{Hom}_{\mathcal{W}_{I_1, L}}(\rho', \rho) \otimes \mathcal{L}_\rho & \text{if } \rho' \in \mathcal{W}_{I_1, L}^\wedge, \\ \bigoplus_{\rho \in \mathcal{W}_{I_2}^\wedge} H^\bullet(\mathbf{P}_1) \otimes \mathrm{Hom}_{\mathcal{W}_{I_2, L}}(\rho', \rho) \otimes \mathcal{L}_\rho & \text{if } \rho' \in \mathcal{W}_{I_2, L}^\wedge, \end{cases}$$

Let $\mathrm{IC}_{\mu'}^L = \mathrm{IC}(\tilde{\mathcal{Y}}_{m(\mu') + 1}^L, \mathcal{L}_{\mu'})[d_{m(\mu') + 1}^L]$. By applying the argument in the proof of Proposition 3.6 to the above formula, we obtain

$$(6.3.8) \quad \begin{aligned} \psi''_*(\mathrm{IC}_{\mu'}^L)[d_n - d_n^L] \\ \simeq \bigoplus_{\mu \in \mathcal{P}_{n, 2}} \mathrm{Hom}_{\tilde{\mathcal{W}}_L}(\tilde{V}_{\mu'}, \tilde{V}_\mu) \otimes \mathrm{IC}(\mathcal{Y}_{m(\mu)}, \mathcal{L}_\mu)[d_{m(\mu)}]. \end{aligned}$$

6.4. Recall the map $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ in 4.1. In this subsection, we follow the notation in 4.3, 4.5. We define varieties

$$\begin{aligned} \tilde{\mathcal{X}}^P &= \{(x, v, gP^\theta) \in G^{\iota\theta} \times V \times H/P^\theta \mid (g^{-1}xg, g^{-1}v) \in \mathcal{X}^P\}, \\ \mathcal{X}^P &= \bigcup_{g \in P^\theta} g(B^{\iota\theta} \times M_n). \end{aligned}$$

We define $\pi' : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{X}}^P$, $\pi'' : \tilde{\mathcal{X}}^P \rightarrow \mathcal{X}$ by $\pi'(x, v, gB^\theta) = (x, v, gP^\theta)$, $\pi''(x, v, gP^\theta) = (x, v)$. Then $\pi = \pi'' \circ \pi'$, and we obtain the following commutative diagram.

$$(6.4.1) \quad \begin{array}{ccccc} \tilde{\mathcal{Y}} & \xrightarrow{\psi'} & \tilde{\mathcal{Y}}^L & \xrightarrow{\psi''} & \mathcal{Y} \\ j_0 \downarrow & & j_1 \downarrow & & \downarrow j_2 \\ \tilde{\mathcal{X}} & \xrightarrow{\pi'} & \tilde{\mathcal{X}}^P & \xrightarrow{\pi''} & \mathcal{X}, \end{array}$$

where j_0, j_2 are natural inclusion maps, and $j_1 : (x, v, g\tilde{L}^\theta) \mapsto (x, v, gP^\theta)$. Put $\tilde{\mathcal{X}}_m^{P,+} = (\pi'')^{-1}\mathcal{X}_m^0$. Then we have a commutative diagram

$$(6.4.2) \quad \begin{array}{ccccc} \tilde{\mathcal{Y}}_m^+ & \xrightarrow{\psi'_m} & \tilde{\mathcal{Y}}_m^{L,+} & \xrightarrow{\psi''_m} & \mathcal{Y}_m^0 \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{X}}_m^+ & \xrightarrow{\pi'_m} & \tilde{\mathcal{X}}_m^{P,+} & \xrightarrow{\pi''_m} & \mathcal{X}_m^0, \end{array}$$

where π'_m is the restriction of π' on $\tilde{\mathcal{X}}_m^+$, π''_m is the restriction of π'' on $\tilde{\mathcal{X}}_m^{P,+}$, and vertical maps are restrictions of the corresponding vertical maps in (6.4.1). Since $\pi_m = \pi''_m \circ \pi'_m$ is proper, π'_m is proper. We note that π''_m is also proper. In fact, since \mathcal{X}^P is the image of $P^\theta \times^{B^\theta} (B^{\iota\theta} \times M_n)$ under the map π , \mathcal{X}^P is closed in \mathcal{X} . Hence $\pi'' : \tilde{\mathcal{X}}^P \simeq H \times^{P^\theta} \mathcal{X}^P \rightarrow \mathcal{X}$ is proper, and so π''_m is proper.

Recall the decomposition $\tilde{\mathcal{X}}_m^+ = \coprod_I \tilde{\mathcal{X}}_I$. We denote by $\tilde{\mathcal{X}}_I^P$ the image of $\tilde{\mathcal{X}}_I$ by π'_m . Then $\tilde{\mathcal{X}}_I^P$ coincides with $\tilde{\mathcal{X}}_{[1,m]}^P$ (resp. $\tilde{\mathcal{X}}_{[2,m+1]}^P$) if $1 \in I$ (resp. $1 \notin I$). Thus $\tilde{\mathcal{X}}_m^{P,+} = \tilde{\mathcal{X}}_{I_1}^P \coprod \tilde{\mathcal{X}}_{I_2}^P$, and $\tilde{\mathcal{Y}}_{I_\varepsilon}^L$ is open dense in $\tilde{\mathcal{X}}_{I_\varepsilon}^P$ for $\varepsilon = 1$ or 2 .

We now show the following formula.

$$(6.4.3) \quad \begin{aligned} (\pi'_m)_* \bar{\mathbf{Q}}_l &\simeq \bigoplus_{\rho' \in \mathcal{W}_{I_1,L}^\wedge} \text{Ind}_{\tilde{\mathcal{W}}_{I_1,L}^L}^{\tilde{\mathcal{W}}_L} (H^\bullet(\mathbf{P}_1^{n-m}) \otimes \rho') \otimes \text{IC}(\tilde{\mathcal{X}}_{I_1}^P, \mathcal{L}_{\rho'}) \\ &\oplus \bigoplus_{\rho' \in \mathcal{W}_{I_2,L}^\wedge} \text{Ind}_{\tilde{\mathcal{W}}_{I_2,L}^L}^{\tilde{\mathcal{W}}_L} (H^\bullet(\mathbf{P}_1^{n-m-1}) \otimes \rho') \otimes \text{IC}(\tilde{\mathcal{X}}_{I_2}^P, \mathcal{L}_{\rho'}). \end{aligned}$$

In fact, since π'_m is proper, by the decomposition theorem, $(\pi'_m)_* \bar{\mathbf{Q}}_l$ can be written as a direct sum of the complexes of the form $A[i]$, where A is a simple perverse sheaf on $\tilde{\mathcal{X}}_m^{P,+}$ with some degree shift i . Note that $(\pi'_m)_* \bar{\mathbf{Q}}_l|_{\tilde{\mathcal{Y}}_m^{L,+}} \simeq (\psi'_m)_* \bar{\mathbf{Q}}_l$, and the decomposition of $(\psi'_m)_* \bar{\mathbf{Q}}_l$ as a semisimple complex is given in (6.3.6). Hence in order to prove (6.4.3), it is enough to show that $\tilde{\mathcal{Y}}_m^{L,+} \cap \text{supp } A \neq \emptyset$ for any direct summand $A[i]$. Since π''_m is proper, again by the decomposition theorem, $(\pi''_m)_* A[i]$ can be written as a direct sum of the complexes $B[j]$, where B is a simple perverse sheaf on \mathcal{X}_m^0 . Since $\text{supp } A$ is irreducible, we may assume that $\text{supp } A$ is contained in $\tilde{\mathcal{X}}_{I_1}^P$ or $\tilde{\mathcal{X}}_{I_2}^P$. Then by the property of ψ''_m , $\dim \text{supp } B < \dim \mathcal{X}_m^0$ if $\tilde{\mathcal{Y}}_m^{L,+} \cap \text{supp } A = \emptyset$. But Proposition 4.8 asserts that $\dim \text{supp } B = \dim \mathcal{X}_m^0$ for any simple component B in $(\pi_m)_* \bar{\mathbf{Q}}_l$. Hence $\tilde{\mathcal{Y}}_m^{L,+} \cap \text{supp } A \neq \emptyset$, and (6.4.3) holds.

The above argument also shows that

$$(6.4.4) \quad \begin{aligned} &(\pi''_m)_* \text{IC}(\tilde{\mathcal{X}}_I^P, \mathcal{L}_{\rho'}) \\ &\simeq \begin{cases} \bigoplus_{\rho \in \mathcal{W}_{I_1}^\wedge} \text{Hom}_{\mathcal{W}_{I_1,L}}(\rho', \rho) \otimes \text{IC}(\mathcal{X}_m^0, \mathcal{L}_\rho) & \text{if } I = I_1, \\ \bigoplus_{\rho \in \mathcal{W}_{I_2}^\wedge} H^\bullet(\mathbf{P}_1) \otimes \text{Hom}_{\mathcal{W}_{I_2,L}}(\rho', \rho) \otimes \text{IC}(\mathcal{X}_m^0, \mathcal{L}_\rho) & \text{if } I = I_2. \end{cases} \end{aligned}$$

Let $\tilde{\mathcal{X}}_m^P$ be the closure of $\tilde{\mathcal{X}}_{[1,m]}^P$ in $\tilde{\mathcal{X}}^P$. Then $\tilde{\mathcal{X}}_m^P = \coprod_{m' < m} \tilde{\mathcal{X}}_{m'}^{P,+} \coprod \tilde{\mathcal{X}}_{[1,m]}^P$ and $\tilde{\mathcal{X}}_m^P \simeq H \times^{P^\theta} \mathcal{X}_m^P$, where $\mathcal{X}_m^P = \bigcup_{g \in P^\theta} g(B^{\iota^\theta} \times M_m)$. (The closure of $\tilde{\mathcal{X}}_{[2,m+1]}^P$ in $\tilde{\mathcal{X}}^P$ is given by $\coprod_{1 \leq m' \leq m} \tilde{\mathcal{X}}_{[2,m'+1]}^P$.) Now by a similar argument as in the last step of Theorem 4.2 (see 4.9), we have the following formulas from (6.3.7) and (6.3.8);

$$(6.4.5) \quad \pi'_* \bar{\mathbf{Q}}_\ell[d_n^L] \simeq \bigoplus_{\mu' \in \mathcal{P}_{n-1,2}} \tilde{V}_{\mu'} \otimes \mathrm{IC}(\tilde{\mathcal{X}}_{m(\mu') + 1}^P, \mathcal{L}_{\mu'})[d_{m(\mu') + 1}^L]$$

$$(6.4.6) \quad \pi''_*(\mathrm{IC}_{\mu'}^P)[a] \simeq \bigoplus_{\mu \in \mathcal{P}_{n,2}} \mathrm{Hom}_{\tilde{W}_L}(\tilde{V}_{\mu'}, \tilde{V}_{\mu}) \otimes \mathrm{IC}(\mathcal{X}_{m(\mu)}, \mathcal{L}_{\mu})[d_{m(\mu)}],$$

where $\mathrm{IC}_{\mu'}^P = \mathrm{IC}(\tilde{\mathcal{X}}_{m(\mu') + 1}^P, \mathcal{L}_{\mu'})$ and $a = d_n - d_n^L + d_{m(\mu') + 1}^L$.

6.5. Let $\tilde{\mathcal{X}}_{\mathrm{uni}}^P = \{(x, v, gP^\theta) \in \tilde{\mathcal{X}}^P \mid x \in G_{\mathrm{uni}}^{\iota^\theta}\}$. Let $\mathcal{O}' = \mathcal{O}_{\mu'}^\bullet$ be the L^θ -orbit in $L_{\mathrm{uni}}^{\iota^\theta} \times V_L$ corresponding to $\tilde{V}_{\mu'}$ under the Springer correspondence for L . We define a variety

$$D = \{(x, v, gP^\theta) \in \tilde{\mathcal{X}}_m^P \mid (g^{-1}xg, g^{-1}v) \in \pi_P^{-1}(\overline{\mathcal{O}'}^\vee)\},$$

where $\pi_P : P_{\mathrm{uni}}^{\iota^\theta} \times V \rightarrow L_{\mathrm{uni}}^{\iota^\theta} \times V_L$ is as in 5.5, and $m = m(\mu') + 1$. Let $\mathrm{IC}_{\mu'}^P$ be as in 6.4. The following fact holds.

$$(6.5.1) \quad \mathrm{supp} \, \mathrm{IC}_{\mu'}^P \cap \tilde{\mathcal{X}}_{\mathrm{uni}}^P \subset D.$$

We show (6.5.1). Let $\mathcal{X}_{m-1}^L = \bigcup_{\ell \in L^\theta} \ell(B_L^{\iota^\theta} \times M_{m-1}^L)$, where $B_L = B \cap L$ is a Borel subgroup of L containing T , and M_{m-1}^L is the image of M_m under the projection $V \rightarrow V_L$. We have a diagram

$$(6.5.2) \quad \tilde{\mathcal{X}}_m^P \xleftarrow{p_1} H \times \mathcal{X}_m^P \xrightarrow{p_2} \mathcal{X}_{m-1}^L,$$

where p_1 is the natural map $H \times \mathcal{X}_m^P \rightarrow \tilde{\mathcal{X}}_m^P \simeq H \times^{P^\theta} \mathcal{X}_m^P$, and p_2 is the restriction to $H \times \mathcal{X}_m^P$ of the map $H \times P^{\iota^\theta} \times V \rightarrow L^{\iota^\theta} \times V_L$, $(g, x, v) \mapsto (\bar{x}, \bar{v})$. (Here $x \rightarrow \bar{x}$ is the natural projection $P^{\iota^\theta} \rightarrow L^{\iota^\theta}$ and $v \mapsto \bar{v}$ is the projection $V \rightarrow V_L$.) Then we have

- (a) p_1 is a principal P^θ -bundle,
- (b) p_2 is a locally trivial fibration with fibre isomorphic to $H \times U_P^{\iota^\theta} \times \mathbf{k}$.

In fact, (a) is clear from the definition. Since M_m is U_P^θ -stable, we have

$$\mathcal{X}_m^P = \bigcup_{\ell \in L^\theta} \ell(B^{\iota^\theta} \times M_m) \simeq U_P^{\iota^\theta} \times \mathbf{k} \times \bigcup_{\ell \in L^\theta} \ell(B_L^{\iota^\theta} \times M_{m-1}^L),$$

and (b) follows.

We consider the complex $\mathrm{IC}_{\mu'}^P$ on $\tilde{\mathcal{X}}_m^P$. By (a), (b), both of p_1, p_2 are smooth maps with connected fibre. It follows that there exists a simple perverse sheaf $A_{\mu'}$ on \mathcal{X}_{m-1}^L such that $p_1^* \mathrm{IC}_{\mu'}^P \simeq p_2^* A_{\mu'}$ up to shift. Note that \mathcal{X}_{m-1}^L is a variety constructed from (L, V_L) defined in a similar way as \mathcal{X}_m from (G, V) . Let $\mathrm{IC}(\mathcal{X}_{m-1}^L, \mathcal{L}_{\mu'}^L)$ be the complex on \mathcal{X}_{m-1}^L appeared in 4.1 with respect to L . We show that

$$(6.5.3) \quad A_{\mu'} \simeq \mathrm{IC}(\mathcal{X}_{m-1}^L, \mathcal{L}_{\mu'}^L)[\dim \mathcal{X}_{m-1}^L].$$

For each $I = [1, m]$ put

$$\begin{aligned} \tilde{\mathcal{Y}}_I^{\natural} &= \tilde{L}^{\theta} \times^{B^{\theta} \cap Z_H(T^{\iota\theta})} (T_{\mathrm{reg}}^{\iota\theta} \times M_I), \\ \hat{\mathcal{Y}}_I^{\natural} &= \tilde{L}^{\theta} \times^{Z_H(T^{\iota\theta})_I} (T_{\mathrm{reg}}^{\iota\theta} \times M_I). \end{aligned}$$

Then $\tilde{\mathcal{Y}}_I \simeq H \times^{\tilde{L}^{\theta}} \tilde{\mathcal{Y}}_I^{\natural}$, $\hat{\mathcal{Y}}_I \simeq H \times^{\tilde{L}^{\theta}} \hat{\mathcal{Y}}_I^{\natural}$. Also we have $\tilde{\mathcal{Y}}_I^L \simeq H \times^{\tilde{L}^{\theta}} \mathcal{Y}_I^L$ by 6.3. We put

$$\begin{aligned} \tilde{\mathcal{Y}}_{L,I} &= L^{\theta} \times^{B_L^{\theta} \cap Z_{L^{\theta}}(T^{\iota\theta})} (T_{\mathrm{reg}}^{\iota\theta} \times M_{I'}^L), \\ \hat{\mathcal{Y}}_{L,I} &= L^{\theta} \times^{Z_{L^{\theta}} L(T^{\iota\theta})_I} (T_{\mathrm{reg}}^{\iota\theta} \times M_{I'}^L), \\ \mathcal{Y}_{L,m-1}^0 &= \bigcup_{\ell \in L^{\theta}} \ell(T_{\mathrm{reg}}^{\iota\theta} \times M_{I'}^L). \end{aligned}$$

where $M_{I'}^L$ is the image of M_I under the natural map $V \rightarrow V_L$ with $I' = [2, m]$, and $Z_{L^{\theta}}(T^{\iota\theta})_I = Z_H(T^{\iota\theta})_I \cap L^{\theta}$. Then we have a commutative diagram

$$(6.5.4) \quad \begin{array}{ccccc} \tilde{\mathcal{Y}}_I & \xleftarrow{\tilde{q}_1} & H \times \tilde{\mathcal{Y}}_I^{\natural} & \xrightarrow{\tilde{q}_2} & \tilde{\mathcal{Y}}_{L,I} \\ \xi_I \downarrow & & \downarrow \alpha_I & & \downarrow \xi_{L,I} \\ \hat{\mathcal{Y}}_I & \xleftarrow{\hat{q}_1} & H \times \hat{\mathcal{Y}}_I^{\natural} & \xrightarrow{\hat{q}_2} & \hat{\mathcal{Y}}_{L,I} \\ \eta'_I \downarrow & & \downarrow \beta_I & & \downarrow \eta_{L,I} \\ \tilde{\mathcal{Y}}_I^L & \xleftarrow{q_1} & H \times \mathcal{Y}_I^L & \xrightarrow{q_2} & \mathcal{Y}_{L,m-1}^0 \end{array}$$

where $\tilde{q}_1, \hat{q}_1, q_1$ and α_I, β_I are defined naturally. \tilde{q}_2 is defined by $(g, \ell * (t, v)) \mapsto \bar{\ell} * (t, \bar{v})$, where $\ell \mapsto \bar{\ell}$ is the projection $\tilde{L}^{\theta} \rightarrow L^{\theta}$, and $v \mapsto \bar{v}$ is the map $M_I \rightarrow M_{I'}^L$, and the maps \hat{q}_2, q_2 are defined similarly. $\xi_{L,I}, \eta_{L,I}$ are defined in a similar way as ξ_I, η_I for $\tilde{\mathcal{Y}}_I, \hat{\mathcal{Y}}_I$. Then $\tilde{q}_1, \hat{q}_1, q_1$ are principal \tilde{L}^{θ} -bundles, and $\tilde{q}_2, \hat{q}_2, q_2$ are principal $H \times \mathbf{k}^*$ -bundles. Moreover, α_I is a \mathbf{P}_1^{n-m} -bundle, and β_I is a finite Galois covering with group $\mathcal{W}_{L,I}$. All the squares in the diagram are cartesian squares.

Note that $\tilde{\mathcal{Y}}_{L,I} \simeq \mathbf{k}^* \times \tilde{\mathcal{Y}}_{I'}^{(n-1)}$, where $\tilde{\mathcal{Y}}_{I'}^{(n-1)}$ is an object for Sp_{2n-2} with $I' = [2, m] \subset [2, n]$, similar to $\tilde{\mathcal{Y}}_I$ for Sp_{2n} . Similar formulas hold for $\hat{\mathcal{Y}}_{L,I}, \mathcal{Y}_{L,m-1}^0$, and the maps ξ_I, η_I are compatible with the situation for Sp_{2n-2} . Thus one can define a local system $\mathcal{L}_{\rho'}^L$ on \mathcal{Y}_{m-1}^0 with respect to Sp_{2n-2} . It follows from the property of

the diagram (6.5.4), one sees that $q_1^* \mathcal{L}_{\rho'} \simeq q_2^* \mathcal{L}_{\rho'}^L$. Note that $\tilde{\mathcal{Y}}_I^L$ is open dense in $\tilde{\mathcal{X}}_m^P$, $\mathcal{Y}_{L,m-1}^0$ is open dense in \mathcal{X}_{m-1}^L , and the lowest row in (6.5.4) is the restriction of the diagram (6.5.2). This shows that the restriction of $A_{\mu'}$ on $\mathcal{Y}_{L,m-1}^0$ gives a local system $\mathcal{L}_{\rho'}^L$ corresponding to μ' . Hence (6.5.3) holds.

Now by the Springer correspondence for L , the support of the restriction of $A_{\mu'}$ on $L_{\text{uni}}^\theta \times V_L$ is contained in $\overline{\mathcal{O}}'$. Then (6.5.1) follows from (6.5.2) and (6.5.3).

6.6. Let $\mathcal{O} = \mathcal{O}_{\mu^\bullet}$ be the H -orbit in $G^\theta \times V$ corresponding to \tilde{V}_μ under the Springer correspondence. Put $d_{\mathcal{O}} = (\dim \mathcal{X}_{\text{uni}} - \dim \mathcal{O})/2$. In view of Theorem 5.4, (5.4.3) implies that $\mathcal{H}^{2d_{\mathcal{O}}}(\pi_* \bar{\mathbf{Q}}_l)|_{\mathcal{O}} \simeq \tilde{V}_\mu \otimes \bar{\mathbf{Q}}_l$, where $\bar{\mathbf{Q}}_l$ is the constant sheaf on \mathcal{O} . Then (6.4.6) implies that

(6.6.1) The inner product $\langle \tilde{V}_\mu, \tilde{V}_{\mu'} \rangle$ coincides with the rank of the constant sheaf $\mathcal{H}^{2d_{\mathcal{O}}+c} \pi_*^P(\text{IC}_{\mu'}^P)|_{\mathcal{O}}$, where $c = d_{m(\mu')+1}^L - d_n^L$.

Then by (6.5.1), we have

(6.6.2) $\langle \tilde{V}_\mu, \tilde{V}_{\mu'} \rangle$ coincides with the rank of the constant sheaf $\mathcal{H}^{2d_{\mathcal{O}}+c}(\pi''|_D)_!(\text{IC}_{\mu'}^P|_D)|_{\mathcal{O}}$.

Let $D^0 = \{(x, v, gP^\theta) \in \tilde{\mathcal{X}}_m^P \mid g^{-1}(x, v) \in \pi_P^{-1}(\mathcal{O}')\}$ be an open subset of D . Let $x_{\mu, \mu'}$ be the rank of the constant sheaf $\mathcal{H}^{2d_{\mathcal{O}}+c}(\pi''|_{D^0})_!(\text{IC}_{\mu'}^P|_{D^0})|_{\mathcal{O}}$. We want to show that

$$(6.6.3) \quad \langle \tilde{V}_\mu, \tilde{V}_{\mu'} \rangle = x_{\mu, \mu'}.$$

First we show that

(6.6.4) The natural map $\mathcal{H}^{2d_{\mathcal{O}}+c}(\pi''|_{D^0})_!(\text{IC}_{\mu'}^P|_{D^0})|_{\mathcal{O}} \rightarrow \mathcal{H}^{2d_{\mathcal{O}}+c}(\pi''|_D)_!(\text{IC}_{\mu'}^P|_D)|_{\mathcal{O}}$ is surjective.

In order to prove (6.6.4), it is enough to show that

$$\mathcal{H}^{2d_{\mathcal{O}}+c}(\pi''|_{D-D^0})_!(\text{IC}_{\mu'}^P|_{D-D^0})|_{\mathcal{O}} = 0,$$

which is equivalent to the statement that

$$\mathbf{H}_c^{2d_{\mathcal{O}}+c}(\pi''^{-1}(z) \cap (D - D^0), \text{IC}_{\mu'}^P) = 0$$

for any $z \in \mathcal{O}$. For any $\mathcal{O}'_1 = \overline{\mathcal{O}'} - \mathcal{O}'$, put $D_{\mathcal{O}'_1} = \{(x, v, gP^\theta) \in \tilde{\mathcal{X}}_m^P \mid g^{-1}(x, v) \in \pi_P^{-1}(\mathcal{O}'_1)\}$. Then $D - D^0$ can be partitioned into locally closed pieces $D_{\mathcal{O}'_1}$. Thus it is enough to show that

$$\mathbf{H}_c^{2d_{\mathcal{O}}+c}(\pi''^{-1}(z) \cap D_{\mathcal{O}'_1}, \text{IC}_{\mu'}^P) = 0$$

for any such a piece \mathcal{O}'_1 . The last equality will follow from the following statement by a hypercohomology spectral sequence.

$$(6.6.5) \quad H_c^i(\pi''^{-1}(z) \cap D_{\mathcal{O}'_1}, \mathcal{H}^j(\text{IC}_{\mu'}^P)) \neq 0 \Rightarrow i + j < 2d_{\mathcal{O}} + c.$$

We show (6.6.5). The hypothesis implies that

$$i \leq 2 \dim(\pi''^{-1}(z) \cap D_{\mathcal{O}'_1}) \leq (\dim \mathcal{X}_{\text{uni}} - \dim \mathcal{O}) - (\dim \mathcal{X}_{\text{uni}}^L - \dim \mathcal{O}'_1)$$

by Proposition 5.6 (ii). It also implies that $\mathcal{H}^j(\text{IC}_{\mu'}^P)|_{D_{\mathcal{O}'_1}} \neq 0$. By (6.5.3), this last condition is equivalent to the condition that $\mathcal{H}^j \text{IC}(\mathcal{X}_{m-1}^L, \mathcal{L}_{\mu'}^L)|_{\mathcal{O}'_1} \neq 0$. By Theorem 5.4 (ii) applied to L , this is equivalent to $\mathcal{H}^j \text{IC}(\overline{\mathcal{O}'}, \bar{\mathbf{Q}}_l)[a]|_{\mathcal{O}'_1} \neq 0$ with $a = \dim \mathcal{O}' - \dim \mathcal{X}_{\text{uni}}^L - \dim \mathcal{X}_{m(\mu')}^L + \dim \mathcal{X}^L$. It follows that

$$j < \dim \mathcal{X}_{\text{uni}}^L - \dim \mathcal{O}'_1 + \dim \mathcal{X}_{m(\mu')}^L - \dim \mathcal{X}^L.$$

This implies (6.6.5) since $\dim \mathcal{X}_{m(\mu')}^L - \dim \mathcal{X}^L = d_{m(\mu')+1}^L - d_n^L$ by (6.5.2). Hence (6.6.4) holds.

Now (6.6.4) implies that $\langle \tilde{V}_\mu, \tilde{V}_{\mu'} \rangle \leq x_{\mu, \mu'}$. Since $\sum_{\mu' \in \mathcal{P}_{n-1,2}} \langle \tilde{V}_\mu, \tilde{V}_{\mu'} \rangle \dim \tilde{V}_{\mu'} = \dim \tilde{V}_\mu$, in order to prove (6.6.3), it is enough to show that

$$(6.6.6) \quad \sum_{\mu' \in \mathcal{P}_{n-1,2}} x_{\mu, \mu'} \dim \tilde{V}_{\mu'} = \dim \tilde{V}_\mu.$$

Put $\delta = (\dim \mathcal{X}_{\text{uni}}^L - \dim \mathcal{O}')/2$. By (6.5.3) and Theorem 5.4 (ii), $\mathcal{H}^j(\text{IC}_{\mu'}^P)|_{D^0} = 0$ unless $j = -a$. This implies that $x_{\mu, \mu'}$ is the rank of the constant sheaf

$$(6.6.7) \quad \mathcal{H}^{2d_{\mathcal{O}}-2\delta}(\pi''|_{D^0})_!(\mathcal{H}^{-a} \text{IC}_{\mu'}^P|_{D^0})|_{\mathcal{O}}$$

since $2d_{\mathcal{O}} + c + a = 2d_{\mathcal{O}} - 2\delta$. It follows from (6.4.5) that $R^{2\delta} \pi'_! \bar{\mathbf{Q}}_l|_{D^0} \simeq \tilde{V}_{\mu'} \otimes (\mathcal{H}^{-a} \text{IC}_{\mu'}^P)|_{D^0}$. This implies that $x_{\mu, \mu'} \dim \tilde{V}_{\mu'}$ is the rank of the constant sheaf

$$R^{2d_{\mathcal{O}}-2\delta}(\pi''|_{D^0})_!(R^{2\delta} \pi'_! \bar{\mathbf{Q}}_l|_{D^0})|_{\mathcal{O}}.$$

By the spectral sequence of the composition $\pi'' \circ \pi'$, the last formula is the same as the constant sheaf $R^{2d_{\mathcal{O}}}(\pi|_{\pi'^{-1}(D^0)})_! \bar{\mathbf{Q}}_l|_{\mathcal{O}}$. If we put $D_{\mu'} = D^0$, $\tilde{\mathcal{X}} = \coprod_{\mu'} \pi'^{-1}(D_{\mu'})$ gives a partition of $\tilde{\mathcal{X}}$ into locally closed pieces. Hence by the long exact sequence associated to $R^* \pi_!$, we see that $\sum_{\mu'} x_{\mu, \mu'} \dim \tilde{V}_{\mu'}$ coincides with the rank of the constant sheaf $R^{2d_{\mathcal{O}}} \pi_* \bar{\mathbf{Q}}_l|_{\mathcal{O}}$, which is equal to $\dim \tilde{V}_\mu$. This proves (6.6.6), and (6.6.3) follows.

6.7. We are now ready to prove Theorem 6.2. Put $\mathcal{U} = \{(z, gP^\theta) \in \mathcal{O} \times H/P^\theta \mid g^{-1}z \in \pi_P^{-1}(\mathcal{O}')\}$, and define $f : \mathcal{U} \rightarrow \mathcal{O}$ by $(z, gP^\theta) \mapsto z$. Then $\mathcal{U} \subset D^0$ and the restriction of π'' on \mathcal{U} coincides with f . It follows from (6.6.7) that $x_{\mu, \mu'}$ coincides with the rank of the constant sheaf $R^{2d_{\mathcal{O}}-2\delta}(\pi''|_{\mathcal{U}})_!(\mathcal{H}^{-a} \text{IC}_{\mu'}^P|_{\mathcal{U}})|_{\mathcal{O}}$. Since $\mathcal{H}^{-a} \text{IC}_{\mu'}^P|_{\mathcal{U}}$ is a constant sheaf $\bar{\mathbf{Q}}_l$, $x_{\mu, \mu'}$ is equal to the rank of the constant sheaf $R^{2d_{\mathcal{O}}-2\delta} f_! \bar{\mathbf{Q}}_l$ on \mathcal{O} , hence is equal to $\dim H_c^{2d_{\mathcal{O}}-2\delta}(f^{-1}(z), \bar{\mathbf{Q}}_l)$ for $z \in \mathcal{O}$. By Proposition 5.6 (ii), we know that $\dim f^{-1}(z) \leq d_{\mathcal{O}} - \delta$ for any $z \in \mathcal{O}$. It follows that $x_{\mu, \mu'}$ is equal to the number of irreducible components in $f^{-1}(z)$ of dimension $d_{\mathcal{O}} - \delta$.

Let $\widehat{U}_z = \{g \in H \mid g^{-1}z \in \pi_P^{-1}(\mathcal{O}')\}$. Since $\widehat{U}_z \rightarrow f^{-1}(z)$ is a principal P^θ -bundle, $x_{\mu, \mu'}$ coincides with the number of irreducible components of \widehat{U}_z of dimension $d_{\mathcal{O}} - \delta + \dim P^\theta$. We define a map $f_1 : \widehat{U}_z \rightarrow \mathcal{O}'$ by $g \mapsto \pi_P(g^{-1}z)$. Then for each $z' \in \mathcal{O}'$, $f_1^{-1}(z') = Y_{z, z'}$ in 6.1. f_1 is L^θ -equivariant with respect to the action of L^θ on \widehat{U}_z by the right multiplication, and on \mathcal{O}' by the conjugation. Hence $x_{\mu, \mu'}$ coincides with the number of irreducible components of $f_1^{-1}(z')$ of dimension

$$d_{\mathcal{O}} - \delta + \dim P^\theta - \dim L^\theta = (\dim Z_H(z) - \dim Z_{L^\theta}(z'))/2 + \dim U_P^\theta.$$

In view of (6.6.3), this completes the proof of Theorem 6.2.

7. DETERMINATION OF SPRINGER CORRESPONDENCE

The explicit description of the Springer correspondence was given in Kato [Ka2], by computing Joseph polynomials. In this section, we shall give an alternate approach based on the restriction theorem (Theorem 6.2).

Theorem 7.1. *Under the notation in Theorem 5.4, we have $\mu^\bullet = \mu$, namely, the Springer correspondence is given by $\mathcal{O}_\mu \leftrightarrow \tilde{V}_\mu$ for each $\mu \in \mathcal{P}_{n,2}$.*

7.2. Before proving the theorem, we need a lemma. First we prepare some notation. Take $\mu = (\mu^{(1)}, \mu^{(2)}) \in \mathcal{P}_{n,2}$, and put $\nu = \mu^{(1)} + \mu^{(2)}$. We write ν as $\nu = (\nu_1, \dots, \nu_m) \in \mathcal{P}_n$ with $\nu_m > 0$. Let us fix integers $m_1, \dots, m_\ell > 0$ such that $\sum_{k=1}^\ell m_k = m$ by the condition that

$$\nu_1 = \dots = \nu_{m_1} > \nu_{m_1+1} = \dots = \nu_{m_1+m_2} > \nu_{m_1+m_2+1} = \dots$$

We denote by $\nu_{[k]}$ the constant value ν_j for $m_1 + \dots + m_{k-1} < j \leq m_1 + \dots + m_k$. For such j , $\mu_j^{(1)}$ is also constant and we denote it by $\mu_{[k]}^{(1)}$.

Take $z = (x, v) \in \mathcal{O}_\mu$, and assume that $x = y\theta(y)^{-1}$ with $y \in A_{\text{uni}}$ and that $v \in M_n$. Then $(y, v) \in A_{\text{uni}} \times M_n$ is of type μ , and by [AH], one can find a Jordan basis $\{v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq \nu_i\}$ of $y - 1$ in M_n such that $(y - 1)v_{i,j} = v_{i,j-1}$ for $j > 1$ and $(y - 1)v_{i,j} = 0$ for $j = 1$, and that $v = \sum_{i=1}^\ell v_{p_i, \mu_{[i]}^{(1)}}$, where $p_i = m_1 + \dots + m_{i-1} + 1$. (Note that in [AH], the normal form for \mathcal{O}_μ is given by (y, v') with $v' = \sum_{i=1}^m v_{i, \mu_i^{(1)}}$. This element is A -conjugate to our (y, v) .) Let $\{v'_{i,j}\}$ be a Jordan basis of $y' = \theta(y)^{-1}$ in M'_n such that $(y' - 1)v'_{i,j} = v'_{i,j+1}$ for $j < \nu_i$ and $(y' - 1)v'_{i,j} = 0$ for $j = \nu_i$, and that $\langle v_{i,j}, v'_{i',j'} \rangle = 0$ unless $i = i', j = j'$.

Let P be the stabilizer of the partial flag $\langle v_{1,1} \rangle \subset \langle v_{1,1} \rangle^\perp$ in G . Then P is an θ -stable parabolic subgroup of G as in 6.1, and H/P^θ is identified with the projective space $\mathbf{P}(V)$. We define $w_i = v_{q_i,1}$ or $w_i = v'_{p_i, \nu_{[i]}}$, where $q_i = m_1 + \dots + m_i$. Then w_i is an x -stable vector in V . Let P_i be the stabilizer of the flag $\langle w_i \rangle \subset \langle w_i \rangle^\perp$ in G . Then P_i is θ -stable, and P_i^θ is the stabilizer of the line $\langle w_i \rangle$ in H . We show the following lemma.

Lemma 7.3. *Under the notation as above,*

$$\dim Z_H(z) \cap P_i = \begin{cases} \dim Z_H(z) - 2q_i + 2 & \text{if } w_i = v_{q_i,1}, \\ \dim Z_H(z) - 2q_i + 1 & \text{if } w_i = v'_{p_i, \nu_{[i]}}. \end{cases}$$

Proof. Put $Q = \{g \in G \mid gv = v, g\langle v \rangle^\perp = \langle v \rangle^\perp, g|_{V/\langle v \rangle^\perp} = \text{id}\}$. Then Q is a θ -stable subgroup of G , and we have $Z_H(z) \cap P_i = Z_{P_i^\theta \cap Q^\theta}(x) = (Z_{P_i \cap Q}(x))^\theta$. Note that the structure of $Z_G(x)$ and $Z_H(x)$ are described as follows (cf. the proof of Proposition 2.3.6 in [BKS]); $Z_G(x) = C \ltimes R$, where C is a subgroup of G generated by g such that

$$\begin{aligned} gv_{i,j} &= \sum_{\substack{k \\ \nu_k = \nu_i}} (a_k v_{k,j} + b_k v'_{k, \nu_i - j + 1}), \\ gv'_{i,j} &= \sum_{\substack{k \\ \nu_k = \nu_i}} (a'_k v_{k, \nu_i - j + 1} + b'_k v'_{k,j}), \end{aligned}$$

and R is the unipotent radical of $Z_G(x)$. Then $C \simeq \prod_{k=1}^\ell GL_{2m_k}$. C and R are θ -stable, and so $Z_H(x) = C^\theta \ltimes R^\theta$. Hence $C^\theta \simeq \prod_{k=1}^\ell Sp_{2m_k}$. Since

$$(7.3.1) \quad Z_{P_i^\theta \cap Q^\theta}(x) = (C \cap P_i \cap Q)^\theta \ltimes (R \cap P_i \cap Q)^\theta,$$

we compute $\dim(C \cap P_i \cap Q)^\theta$ and $\dim(R \cap P_i \cap Q)^\theta$ separately. For $k = 1, \dots, \ell$, let V_k be the symplectic space with $\dim V_k = 2m_k$. We fix a symplectic basis $e_1^{(k)}, \dots, e_{m_k}^{(k)}, f_1^{(k)}, \dots, f_{m_k}^{(k)}$ of V_k . Let C_k, D_k be the subgroups of $GL(V_k) \simeq GL_{2m_k}$ defined by

$$\begin{aligned} C_k &= \{g \in GL(V_k) \mid gv_0 = v_0, g\langle v_0 \rangle^\perp = \langle v_0 \rangle^\perp, g|_{V_k/\langle v_0 \rangle^\perp} = \text{id}\}, \\ D_k &= \{g \in C_k \mid g\langle w_0 \rangle = \langle w_0 \rangle, g\langle w_0 \rangle^\perp = \langle w_0 \rangle^\perp\} \end{aligned}$$

where $v_0 = e_1^{(k)}$, and $w_0 = e_{m_k}^{(k)}$ or $w_0 = f_1^{(k)}$. Note that $D_k = C_k$ if $v_0 = w_0$. We denote by θ the involution on GL_{2m_k} defined in a similar way as the case of G . Then C_k, D_k are θ -stable, and we have

$$(7.3.2) \quad (C \cap P_i \cap Q)^\theta \simeq D_i^\theta \times \prod_{k \neq i} C_k^\theta$$

Since $C_k^\theta \simeq (\{1\} \times Sp_{2m_k-2}) \ltimes U_1$, where U_1 is the unipotent radical of a parabolic subgroup of Sp_{2m_k} whose Levi subgroup is isomorphic to $GL_1 \times Sp_{2m_k-2}$. It follows that

$$(7.3.3) \quad \dim C_k^\theta = 2m_k^2 - m_k.$$

Next we show that

$$(7.3.4) \quad \dim D_k^\theta = \begin{cases} 2m_k^2 - 3m_k + 2 & \text{if } w_0 = e_{m_k}^{(k)} \text{ and } m_k \geq 2, \\ 2m_k^2 - 3m_k + 1 & \text{if } w_0 = f_1^{(k)}. \end{cases}$$

First assume that $w_0 = e_{m_k}^{(k)}$ and $m_k \geq 2$. Then $v_0 \in \langle w_0 \rangle^\perp$. We define a new basis h_1, \dots, h_{2m_k} of V_k as follows;

$$h_j = \begin{cases} v_0 & j = 1, \\ w_0 & j = 2, \\ e_{j-1}^{(k)} & 3 \leq j \leq m_k, \\ f_{j+1-m_k}^{(k)} & m_k + 1 \leq j \leq 2m_k - 1, \\ f_1^{(k)} & j = 2m_k. \end{cases}$$

Then we have

$$\begin{aligned} \langle v_0 \rangle^\perp &= \langle h_1, \dots, h_{2m_k-1} \rangle, \\ \langle w_0 \rangle^\perp &= \langle h_1, \dots, h_{2m_k-2}, h_{2m_k} \rangle, \\ \langle w_0 \rangle^\perp \cap \langle v_0 \rangle^\perp &= \langle h_1, \dots, h_{2m_k-2} \rangle, \end{aligned}$$

and the condition for $g \in D_k$ is given by

$$\begin{aligned} gh_1 &= h_1, \\ g\langle h_2 \rangle &= \langle h_2 \rangle, \\ g\langle h_1 \dots h_{2m_k-2} \rangle &= \langle h_1, \dots, h_{2m_k-2} \rangle, \\ g\langle h_1, \dots, h_{2m_k-2}, h_{2m_k} \rangle &= \langle h_1, \dots, h_{2m_k-2}, h_{2m_k} \rangle, \\ gh_{2m_k} &\in h_{2m_k} + \langle h_1, \dots, h_{2m_k-1} \rangle. \end{aligned}$$

We have $D_k^\theta \simeq (GL_1 \times Sp_{2m_k-4}) \ltimes U_2$, where U_2 is the unipotent radical of a parabolic subgroup of Sp_{2m_k} whose Levi subgroup is isomorphic to $GL_2 \times Sp_{2m_k-4}$. The first formula in (7.3.4) follows from this.

Next assume that $w_0 = f_1^{(k)}$. Then $v_0 \notin \langle w_0 \rangle^\perp$. We define a new basis h_1, \dots, h_{2m_k} as follows;

$$h_j = \begin{cases} v_0 & j = 1, \\ e_j^{(k)} & 2 \leq j \leq m_k, \\ f_{j+1-m_k}^{(k)} & m_k + 1 \leq j \leq 2m_k - 1, \\ w_0 & j = 2m_k \end{cases}$$

Then we have

$$\begin{aligned} \langle v_0 \rangle^\perp &= \langle h_1, \dots, h_{2m_k-1} \rangle, \\ \langle w_0 \rangle^\perp &= \langle h_2, \dots, h_{2m_k} \rangle, \end{aligned}$$

$$\langle v_0 \rangle^\perp \cap \langle w_0 \rangle^\perp = \langle h_2, \dots, h_{2m_k-1} \rangle,$$

and the condition for $g \in D_k$ is given by

$$\begin{aligned} gh_1 &= h_1, \\ g\langle h_2, \dots, h_{2m_k-1} \rangle &= \langle h_2, \dots, h_{2m_k-1} \rangle, \\ gh_{2m_k} &= h_{2m_k}. \end{aligned}$$

Then $D_k^\theta \simeq Sp_{2m_k-2}$, and the second formula in (7.3.4) follows.

Now (7.3.2) implies, by (7.3.3) and (7.3.4), that

$$\begin{aligned} (7.3.5) \quad \dim(C \cap P_i \cap Q)^\theta &= \sum_{k \neq i} (2m_k^2 - m_k) + (2m_i^2 - 3m_i + 2 - \varepsilon) \\ &= \dim C^\theta - 2m - 2m_i + 2 - \varepsilon, \end{aligned}$$

where $\varepsilon = 0$ (resp. $\varepsilon = 1$) if $w_i = v_{q_i,1}$ (resp. $w_i = v'_{p_i, \nu_{[i]}}$). Note that this formula holds even if $m_i = 1$ and $\varepsilon = 1$.

Next we show that

$$(7.3.6) \quad \dim(R \cap P_i \cap Q)^\theta = \dim R^\theta - 2|\mu^{(1)}| + 2m - 2 \sum_{k=1}^{i-1} m_k.$$

For each i , choose a pair (i', j') such that $i' > i$ or $i' \leq i, j' < \nu_i$. Then the assignment $v_{i,j} \mapsto v_{i,j} + c_{i,i',j'} v'_{i',j'}$ ($c_{i,i',j'} \in \mathbf{k}$) gives rise to a unique element in R , which is denoted by $g(c_{i,i',j'})$. Then $\prod_{i,i',j'} g(c_{i,i',j'}) \leftrightarrow (c_{i,i',j'})_{i,i',j'}$ gives an isomorphism between R and an affine space \mathbf{A}^c , where $c = \dim R$ (the product is taken under a suitable order). Then for $g \in R$, the condition $gv = v$ is given by a system of linear equations with respect to the variables $(c_{i,i',j'})$, one equation for each $v_{i,j}$ such that $j < \mu_i^{(1)}$, and for each $v'_{i,j}$ such that $\nu_i - j < \mu_i^{(1)}$. It follows that the number of such equations is equal to $\sum_{k=1}^{\ell} 2m_k(\mu_{[k]}^{(1)} - 1)$, and they are linearly independent. On the other hand, for $g \in R$, the condition $g\langle v \rangle^\perp = \langle v \rangle^\perp$ is given by a system of linear equations, one equation for each $v_{i,j}$ such that $\nu_i - j < \mu_i^{(1)}$, and for each $v'_{i,j}$ such that $j < \mu_i^{(1)}$, together with $\ell - 1$ linear equations arising from the $\ell - 1$ dimensional space $\langle v'_{p_i, \mu_{[i]}^{(1)}} \mid 1 \leq i \leq \ell \rangle \cap \langle v \rangle^\perp$. (The condition $g|_{V/\langle v \rangle^\perp} = \text{id}$ is then automatically satisfied since $g \in R$.) Hence the number of equations is given by $\sum_{k=1}^{\ell} 2m_k(\mu_{[k]}^{(1)} - 1) + (\ell - 1)$. By a similar argument, the condition for $g \in R$ to be $g\langle w_i \rangle = \langle w_i \rangle$ (resp. $g\langle w_i \rangle^\perp = \langle w_i \rangle^\perp$) is both given by a system of linear equations whose number is equal to $2m_1 + \dots + 2m_{i-1}$. Since all of those linear equations are linearly independent, we see that

$$\dim(R \cap P_i \cap Q) = \dim R - 4|\mu^{(1)}| + 4m - (\ell - 1) - 4 \sum_{k=1}^{i-1} m_k.$$

If we pass to $(R \cap P_i \cap Q)^\theta$, the linear equation corresponding to $v_{i,j}$ with $j < \mu_i^{(1)}$ and the equation corresponding to $v'_{i,j}$ with $j < \mu_i^{(1)}$ gives one equation, and similarly for $v_{i,j}$ and $v'_{i,j}$ with $\nu_i - j < \mu_i^{(1)}$. This phenomenon also occurs for the linear equations with respect to P_i , and we obtain the half of those linear equations. However, the $(\ell - 1)$ linear equations does not give a restriction on R^θ . Hence we obtain (7.3.6).

The lemma now follows from (7.3.5) and (7.3.6), if we notice that $\dim Z_H(z) = \dim Z_H(x) - 2|\mu^{(1)}|$ (see the proof of Lemma 2.3). \square

7.4. Take $\mu \in \mathcal{P}_{n,2}$ with $\nu = \mu^{(1)} + \mu^{(2)}$. We follow the notation in 7.2. Let $\mu' = (\mu'^{(1)}, \mu'^{(2)}) \in \mathcal{P}_{n-1,2}$ be such that

(i) the Young diagram $\mu'^{(1)}$ is obtained from the Young diagram $\mu^{(1)}$ by removing one node, or

(ii) $\mu'^{(2)}$ is obtained from $\mu^{(2)}$ by removing one node.

We assume that the removing node is contained in q_i -th row in each case, i.e., $\mu'^{(1)}_{q_i} = \mu^{(1)}_{q_i} - 1$ or $\mu'^{(2)}_{q_i} = \mu^{(2)}_{q_i} - 1$. Let $d_\mu = \nu_H - \dim \mathcal{O}_\mu / 2$ and $d_{\mu'} = \nu_{H'} - \dim \mathcal{O}_{\mu'} / 2$, where $H' = Sp_{2n-2}$. We note that

$$(7.4.1) \quad d_\mu - d_{\mu'} = \begin{cases} 2q_i - 2 & \text{case (i),} \\ 2q_i - 1 & \text{case (ii).} \end{cases}$$

In fact, by Lemma 2.3, we have $d_\mu = n + 2n(\mu) - |\mu^{(1)}|$, and a similar formula holds for $d_{\mu'}$. Since $|\mu^{(1)}| - |\mu'^{(1)}| = 1$ or 0 according to the case (i) or (ii), and $n(\mu) - n(\mu') = q_i - 1$, we obtain (7.4.1).

Take $z = (x, v) \in \mathcal{O}_\mu$ as in 7.2, and put $w_i = v_{q_i,1}$ in case (i), and $w_i = v'_{p_i, \nu_{[i]}}$ in case (ii). Let x' be the linear transformation on the symplectic space $\overline{V}_i = \langle w_i \rangle^\perp / \langle w_i \rangle$. We have $v \in \langle w_i \rangle^\perp$ (note that $\mu^{(2)}_{q_i} \neq 0$ in case (ii)), and let v' be the image of v on \overline{V}_i . Then one can check that $z' = (x', v') \in \mathcal{O}_{\mu'}$.

We are now ready to prove the theorem. We consider the variety $f^{-1}(z)$ appeared in 6.7 instead of considering the variety $Y_{z,z'}$ in 6.1. Let $\mathcal{U}_z = \{gP^\theta \in H/P^\theta \mid g^{-1}z \in \pi_P^{-1}(\mathcal{O}_{\mu'})\}$ which is isomorphic to $f^{-1}(z)$ for $\mathcal{O}' = \mathcal{O}_{\mu'}$. Let P_i be the stabilizer of the flag $\langle w_i \rangle \subset \langle w_i \rangle^\perp$ in G . Then $g_i P^\theta \in \mathcal{U}_z$ for $P_i = g_i P g_i^{-1}$. $Z_H(z)$ acts on \mathcal{U}_z from the left, and we consider the $Z_H(z)$ orbit Y_i of $g_i P^\theta$ in \mathcal{U}_z . By Lemma 7.3 and (7.4.1), we have

$$(7.4.2) \quad \dim Y_i = \dim Z_H(z) - \dim(Z_H(z) \cap P_i) = d_\mu - d_{\mu'}.$$

By 6.7 (an equivalent form of Theorem 6.2), all the irreducible components in \mathcal{U}_z have dimension $\leq d_\mu - d_{\mu'}$, and the number of irreducible components of \mathcal{U}_z of dimension $d_\mu - d_{\mu'}$ coincides with the multiplicity $\langle \rho_\mu, \rho_{\mu'} \rangle$. By (7.4.2), the closure of Y_i gives an irreducible component of \mathcal{U}_z of the required dimension. It follows that $\langle \rho_\mu, \rho_{\mu'} \rangle \geq 1$. Since it is known that the restriction of an irreducible representation of W_n to W_{n-1} is multiplicity free, we see that $\langle \rho_\mu, \rho_{\mu'} \rangle = 1$. By induction on the rank of G , we may assume that $\rho_{\mu'} \simeq \tilde{V}_{\mu'}$. Then our result shows that the restriction of

ρ_μ on W_{n-1} coincides with that of \tilde{V}_μ . Since the irreducible representation of W_n is determined completely by its restriction on W_{n-1} if $n \geq 3$ or if its degree ≥ 2 , we see that $\rho_\mu \simeq \tilde{V}_\mu$ in this case. Hence we may assume that $n = 2$, and we have only to distinguish ρ_μ for $\mu = ((2), -), ((1^2), -)$ and for $\mu = (-, (2)), (-, (1^2))$. But by Lemma 5.2, we already know that $\rho_\mu = \tilde{V}_\mu$ for $\mu = ((1^2), -), (-, (1^2))$, and so the remaining cases are determined. This completes the proof of Theorem 7.1.

Remark 7.5. By making use of Theorem 5.14, one can show that the correspondence $\lambda \mapsto \lambda^\bullet$ for $\lambda \in \mathcal{P}_n$ in the formula (1.14.2) in Proposition 1.14 is identical, i.e., for the map $\pi'_1 : \tilde{G}_{\text{uni}}^{\iota\theta} \rightarrow G_{\text{uni}}^{\iota\theta}$ (the map π_1 in 1.10. We changed the notation to distinguish this with $\pi_1 : \tilde{\mathcal{X}}_{\text{uni}} \rightarrow \mathcal{X}_{\text{uni}}$), we have

$$(5.18.1) \quad (\pi'_1)_* \bar{\mathbf{Q}}_l[\dim G_{\text{uni}}^{\iota\theta}] \simeq H^\bullet(\mathbf{P}_1^n) \otimes \bigoplus_{\lambda \in \mathcal{P}_n} V_\lambda \otimes \text{IC}(\bar{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_\lambda].$$

In fact, since $G^{\iota\theta} \simeq G^{\iota\theta} \times \{0\}$, $\tilde{G}^{\iota\theta} \simeq \tilde{\mathcal{X}}_m^+$ for $m = 0$ under the notation of Section 4, $(\pi'_1)_* \bar{\mathbf{Q}}_l$ coincides with the restriction of $(\pi_m)_* \bar{\mathbf{Q}}_l = (\bar{\pi}_m)_* \bar{\mathbf{Q}}_l$ (for $m = 0$) to \mathcal{X}_{uni} up to shift. Here by Proposition 4.8 (or rather by (4.9.1)), we have

$$(5.18.2) \quad (\bar{\pi}_0)_* \bar{\mathbf{Q}}_l \simeq \bigoplus_{\lambda \in \mathcal{P}_n} H^\bullet(\mathbf{P}_1^n) \otimes V_\lambda \otimes \text{IC}(\mathcal{X}_0, \mathcal{L}_\lambda).$$

By Theorem 5.4 (ii), together with Theorem 7.1, we see that

$$(5.18.3) \quad \text{IC}(\mathcal{X}_0, \mathcal{L}_\lambda)|_{\mathcal{X}_{\text{uni}}} \simeq \text{IC}(\bar{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)$$

up to shift. (Here we identify $\mathcal{O}_\lambda \subset G_{\text{uni}}^{\iota\theta}$ with $\mathcal{O}_{(-;\lambda)} \subset \mathcal{X}_{\text{uni}}$ under the closed embedding $G_{\text{uni}}^{\iota\theta} \simeq G_{\text{uni}}^{\iota\theta} \times \{0\} \hookrightarrow \mathcal{X}_{\text{uni}}$.) (5.18.1) follows from (5.18.2) and (5.18.3).

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